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MODULAR FACE LATTICES: LOW DIMENSIONAL CASES

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ABSTRACT. Let K be a self dual cone with a modular lattice of faces. If $\dim K = 4$, then K is strictly convex. If $\dim K = 5$, then either K is strictly convex or every maximal face is of dimension 3. An example is given of a self dual cone K which has an orthomodular but not modular lattice of faces.

The notations and conventions are those of **[2]** and **[3]**. Recall that cone K in a real vector space V is a subset such that if $x, y \in K$, $\alpha, \beta \geqq 0$, then $\alpha x + \beta y \in K$. The cones considered here will be topologically closed, pointed ($K \cap (-K) = \{0\}$), and full (non-empty interior). Also V is assumed to be finite dimensional. K defines an order in V by $x \geqq 0$ if and only if $x \in K$ (cf. **[1]**). We shall write: $x \geqq y$ if $x - y \in K$; $x > y$ if $x \geqq y$ and $x \neq y$; and $x \gg y$ if $x - y \in \operatorname{int} K$. A subset F of K is a *face* if and only if $0 \leqq x \leqq y$ and $y \in F$ implies $x \in F$. Let $\mathscr{F}(K)$ denote the set of all faces of K, and if $S \subset K$, put $\varphi(S) = \bigcap\{F: F \in \mathscr{F}(K) \text{ and } F \supset S\}$. Then $\mathscr{F}(K)$, ordered by inclusion, becomes a lattice if we define $F \vee G = \varphi(F \cup G)$, $F, G \in \mathscr{F}(K)$, and $F \wedge G = F \cap G$.

If $p \in K$ and $\varphi(p)$ is a ray, we call p an *extremal* and let Ext K denote the set of all extremals. If $F \in \mathscr{F}(K)$, we shall also write $F \lhd K$. More generally, since any face is full in its span, we may write $F \lhd G$ if F, G are faces of K and $F \subseteqq G$ (cf. **[2]**). Let $\langle F \rangle = F - F$ denote the linear span of F. We set $\dim F = \dim \langle F \rangle$. If $\mathscr{F}(K)$ is modular, then any two maximal chains linking $\{0\}$ and a face F will have the same number of elements. If there are $k + 1$ elements in a maximal chain between $\{0\}$ and F, we call k the *height* of F and write $h(F) = k$. (In the lattice theory this number is often called the dimension, but we wish to use the latter term for the algebraic dimension.) Note that if $F \in \mathscr{F}(K)$, then $h(F) \leqq \dim F$, and in general equality holds only when F is an atom or K is simplicial. If $h(K) = 2$, then either K is strictly convex or a two dimensional simplicial cone. More generally as theorem 2 of **[3]** and the following lemma show, it is enough to consider only indecomposable cones K. Recall that a cone K is *decomposable* (cf. **[6]**) if there are proper subsets $K_1, K_2 \subset K$ such that

(1) $\forall x \in K, \exists x_i \in K_i$ such that $x = x_1 + x_2$,

(2) span $K_1 \cap$ span $K_2 = \{0\}$

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