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THE MODULAR FUNCTION AND THE MODULUS OF A DOUBLY-CONNECTED REGION

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ABSTRACT. Let D be a doubly connected region and let $K = K(z, \bar{z})$, $z \in D$, be its Szegö kernel. One then forms the conformal invariant $J(z) = K^{-2}\partial_z\bar{\partial}_z \log K$ and also finds $\alpha_D = \max_{z \in D} J(z)$. For $\beta = \beta_D = 4\pi^2/\alpha_D$, one has $\beta \in (0, 1)$. Let Mod $D \in (1, \infty)$ be the modulus of D. Then

$$(\text{Mod } D)^{-1} = \sum_{n=0}^{\infty} \frac{\delta_n}{2^{4n+1}} \left[\frac{1-(1-\beta)^{1/4}}{1+(1-\beta)^{1/4}}\right]^{4n+1},$$

where $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of positive integers arising from the inversion of the modular function; thus $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150, \ldots$. The series converges rapidly and usually the first two terms suffice. A truncation error analysis is provided.

1. **Introduction.** Our recent work [3] conceals in it a rather interesting relationship between the modular function, the analytic capacity and the modulus of a doubly-connected region. This relationship may be exploited to yield an efficient method for determining the modulus of a doubly-connected region. Basically, this relationship can be described in the following way. Let D be a doubly-connected region with no degenerate boundary component and let $C(z)$ be its analytic capacity at $z \in D$ (see definition below). One then forms the well-defined conformal invariant

$$J(z) = \pi^2 C^{-2} \Delta \log C, \quad C = C(z)$$

where Δ denotes the usual Laplace operator $\Delta = 4\partial_z\bar{\partial}_z$. We shall use the fact (see [3]) that $J(z) \geqq 4\pi^2$ for all $z \in D$ and that, within a proper approach, $J(z) = 4\pi^2$ for $z \in \partial D$. We define $\alpha_D = \max_{z \in D} J(z)$, $\beta_D = 4\pi^2/\alpha_D$ and thus $\beta \equiv \beta_D \in (0, 1)$. Let r^{-1} $(0 < r < 1)$ be the modulus of D. Then

$$r = \sum_{n=0}^{\infty} \frac{\delta_n}{2^{4n+1}} \left[\frac{1-(1-\beta)^{1/4}}{1+(1-\beta)^{1/4}}\right]^{4n+1}, \tag{1.1}$$

where $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of positive integers arising from the well-known inversion of the modular function (see Weierstrass [10, p. 276]). Thus $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150$, $\delta_4 = 1{,}707$, $\delta_5 = 20{,}910$,

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