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RINGS WITH INVOLUTION AS PARTIALLY ORDERED ABELIAN GROUPS

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Let $(S, *)$ be a ring with involution $*$. The involution is *positive definite* if, for all finite subsets $\{r_i\}$ of S, $\sum r_i r_i* = 0$ implies all the r_i are zero. Then the set of self-adjoint elements of S, denoted S^s, possesses a natural partial ordering, with positive cone consisting of elements of the form $\sum r_i r_i*$; with this ordering, S^s is a directed partially ordered abelian group. Let S_b denote the set of bounded elements, that is, the set of elements s such that $ss*$ is less than an integral multiple of 1 in this ordering. Then S_b is a $*$-subalgebra of S whenever S is an algebra over the rationals. We will be studying the objects S_b, S^s, and $(S_b)^s$, from the point of view of their ordered structures.

For instance, suppose S is a field, and $*$ is the identity. Then S is formally real, and S_b must be Prüfer domain, all of whose residue fields are themselves formally real (and in fact, are embeddable in the reals). Viewing S_b as a partially ordered abelian group with order unit 1 (indeed, S_b is the convex subgroup of S generated by 1), S_b has the Riesz decomposition property, and its normalized extremal states are precisely the ring homomorphisms into the reals. There is a natural mapping from the collection of total orderings of S to the set of extremal states of S_b, and this in turn maps to Spec S_b (the prime ideal space of S_b); when S is even a real algebra much more can be said.

If either S is a field and $*$ is not the identity, or S is a quaternionic division algebra with the natural involution, essentially the same properties hold, with the appropriate modifications. A useful tool here is an involutory version of the Artin Schreier Theorem, about the existence of sufficiently many total orderings finer than the natural ordering.

Studies are made of several specific bounded subrings. For instance, if S is the rational function field in one variable over the reals, then S_b is a Dedekind domain with class group of order 2, with spectrum the unit circle (in the point-open topology), and all of its maximal ideals are not principal.

Expanding the scope of S somewhat, we next allow S to be a division

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