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ON THEOREMS OF B. H. NEUMANN CONCERNING *FC*-GROUPS

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1. **Introduction**. The theorems we are concerned with here are the characterizations of central-by-finite groups and finite-by-abelian groups given by B. H. Neumann [6]. He proved that a group G is central-by-finite if and only if each subgroup has only finitely many conjugates or, equivalently, U/U_G is finite for each subgroup U of G. Here U_G denotes the core of U, that is, the largest normal subgroup of G contained in U; we use U^G to denote the normal closure of U in G. The "dual" characterization given by Neumann was that G is finite-by-abelian if and only if $|U^G:U|$ is finite for each subgroup U of G.

It was indicated by Eremin [3] that it is only necessary to consider the abelian subgroups of G in the first of these theorems. Although one of the apparent simplifications in Eremin's proof is incorrect, his strategy of concentrating on direct products of cyclic groups does give a slightly simpler proof of Neumann's results and we use this in the generalizations that we give here.

Our main concern is to consider FC-groups in which $|G'| < \mathfrak{m}$ or $|G/Z| < \mathfrak{m}$; here as throughout the paper $\mathfrak{m}$ denotes an infinite cardianl. We prove the following theorem.

THEOREM A. *Let G be an FC-group. Then $|G'| < \mathfrak{m}$ if and only if $|U^G: U| < \mathfrak{m}$ for each $U \leqq G$.*

The results for G/Z cannot be proved for all FC-groups but hold in large subclasses. We define $\mathfrak{Z}_\mathfrak{m}$ to be the class of FC-groups in which $|G: C_G(U)| < \mathfrak{m}$ whenever U is generated by fewer than $\mathfrak{m}$ elements. [If G is periodic or $\mathfrak{m}$ is uncountable, U being generated by fewer than $\mathfrak{m}$ elements simply means $|U| < \mathfrak{m}$]. In [9], we defined $\mathfrak{Z}$ to be the class of locally finite groups G satisfying the condition: if $\mathfrak{m}$ is an infinite cardinal and $H \leqq G$ such that $|H| < \mathfrak{m}$, then $|G: C_G(H)| < \mathfrak{m}$. It is clear that $\mathfrak{Z} \subseteqq \mathfrak{Z}_\mathfrak{m}$ for each $\mathfrak{m}$ and all the evidence we have suggests that $\mathfrak{Z}$ is a very large subclass of the class of periodic FC-groups. It should also be noted that if $\mathfrak{m} = \aleph_0$, then $\mathfrak{Z}_\mathfrak{m}$ is the class of all FC-groups and so Neumann's result is a special case of the following theorem.

THEOREM B. *Let $G \in \mathfrak{Z}_\mathfrak{m}$. Then the following are equivalent:*

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