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PERIODIC, NONPERIODIC AND IRREGULAR MOTIONS IN A HAMILTONIAN SYSTEM

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ABSTRACT. We study orbit structures in the neighborhoods of homoclinic orbits connecting saddlepoints with real eigenvalues in certain four-dimensional Hamiltonian systems. Under suitable hypotheses we prove the existence of dense non-periodic orbits and of periodic orbits of infinitely many periods.

1. **Introduction.** Recently Markus and Meyer [**12**] have proved the generic existence of solenoids in Hamiltonian systems of dimension $\geqq 4$. The existence of solenoids implies that the set of orbits of the system is extremely complicated and in particular, that it contains orbits of arbitrary periods in addition to nonperiodic recurrent motions. The results of [**12**] are general and somewhat abstract. Devaney [**4–7**], has obtained more specific results on Hamiltonian systems possessing homoclinic orbits connecting hyperbolic saddle points, cf. Silnikov's studies of homoclinic orbits in general (non-Hamiltonian) system [**15, 16**].

Here we discuss a specific situation applicable to certain mechanical problems, e.g., the erratic behavior of a spherical pendulum swinging over arrays of two or three magnets. We shall study the structure of orbits in the neighborhood of solutions doubly asymptotic to a saddle point. Such doubly asymptotic solutions are called homoclinic orbits and occur generically in Hamiltonian systems [**4, 5**]. Their occurrence in non-Hamiltonian differential equations is, of course, non-generic, cf., [**2**].

In a four dimensional system there are essentially three structurally stable types of saddle point possible: the saddle-center, with eigenvalues $\{\pm\alpha, \pm i\beta | \alpha, \beta > 0\}$; the "saddle-saddle", with eigenvalues $\{\pm l, \pm k | l, k > 0\}$ and the saddle-focus, with eigenvalues $\{\pm(\alpha \pm i\beta) \mid \alpha, \beta > 0\}$. The saddle foucs was treated in [**4**]. Here we consider the saddle-saddle and briefly mention the saddle-center. Only the saddle-focus and saddle-saddle are hyperbolic fixed points, since the saddle-center has a pair of purely imaginary eigenvalues.

Since the saddle-saddle is a hyperbolic critical point with stable and unstable manifolds M^s, M^u each of dimension 2, transverse homoclinic orbits can occur generically at the (transverse) intersection of $M^s(o)$ and $M^u(o)$ within the three dimensional energy surface $H^{-1}(0)$, where o is

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