ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

SINGULAR NONLINEAR EVOLUTION EQUATIONS

R. E. SHOWALTER*

ABSTRACT. Sufficient conditions are given to obtain existence and uniqueness of strong solutions to $u'(t) + A(u(t)) \ni f(t)$ on $(-\infty, 0)$ where A is a maximal monotone operator in Hilbert space. Applications to certain nonlinear problems for partial differential equations are described.

1. **Introduction.** We shall consider nonlinear evolution equations of the form

$$\frac{du(t)}{dt} + \mu u(t) + A(u(t)) \ni f(t), \quad -\infty < t < 0, \tag{1.1}$$

in a Hilbert space $\mathbf{H}$, where $\mu \in \mathbf{R}$, the real numbers, and A is a maximal monotone operator in $\mathbf{H}$ [2]. The solution will be obtained in the Hilbert space $\mathscr{H}_\omega$ of functions u: $(-\infty, 0) \to \mathbf{H}$ which are square-summable with the measure $e^{-2\omega t}\,dt$ for appropriate $\omega \in \mathbf{R}$. That is, $u \in W_w^{1,2}((-\infty, 0), \mathbf{H})$, the class of functions u in $\mathscr{H}_\omega$ whose (strong) derivatives u' belong to $\mathscr{H}_\omega$.

We first show that the linear operator "$(d/dt) + \mu$" is maximal monotone on $\mathscr{H}_\omega$ when $\mu + \omega \geqq 0$. Then we obtain

THEOREM 1. *Let A be maximal monotone in* $\mathbf{H}$, $A(0) \ni 0$ *and* $\omega + \mu > 0$. *For each $f \in W_\omega^{1,2}((-\infty, 0), \mathbf{H})$ there exists a unique solution $u \in W_\omega^{1,2}((-\infty, 0), \mathbf{H})$ of* (1.1).

For a restricted class of maximal montone operators, the subdifferentials, we obtain a corresponding result. Let φ: $\mathbf{H} \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The operator on $\mathbf{H}$ defined by

$$\partial\varphi(u) \equiv \{f \in \mathbf{H}: (f, v - u)_H \leqq \varphi(v) - \varphi(u) \text{ for all } v \in \mathbf{H}\}$$

is a maximal monotone $\partial\varphi$ called the subdifferential of φ [2].

THEOREM 2. *Let φ: $\mathbf{H} \to [0, +\infty]$ be convex and lower semicontinuous with $\varphi(u_0) = 0$ for some $u_0 \in \mathbf{H}$. The operator "$(d/dt) + \mu + \partial\varphi$" is maximal monotone on $\mathscr{H}_\omega$ in each of the following situations*: (a) $\mu \geqq 0$, $2\omega + \mu \geqq 0$ *and one of* $\mu = 0$ *or* $\varphi(0) = 0$ *or* $\omega < 0$; (b) $\mu < 0$, *there is a* $p \geqq 2$ *such that* $\varphi(\lambda u) \leqq \lambda^p \varphi(u)$ *for all* $\lambda \geqq 1$, *and* $2\omega + p\mu > 0$. *If, in*

*This work was supported in part by a National Science Foundation grant.
Received by the editors on October 4, 1978, and in revised form January 29, 1979.