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# LINEAR SYMPLECTIC STRUCTURES ON BANACH SPACES

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ABSTRACT. In recent years, both J. Marsden and A. Weinstein have called attention to the practical and theoretical importance of symplectic forms on Banach manifolds. In particular, Weintein's proof of the Darboux theorem in infinite dimensions reduces the local classification of symplectic structures on a Banach manifold to the study of linear (i.e., "constant") symplectic forms on a Banach space.

In previous work, the author has proved a version of the Morse index theorem for partial differential equations, using a topological approach based on symplectic forms in Hilbert space.

In the present paper, we attempt a more systematic study of the linear theory. The work is expository but includes several new results and examples. We prove in §*1*, for example, that the pull-back action of the general linear group on symplectic forms is *stable*; i.e., the orbits are open sets.

We define isotropic and lagrangian subspaces and give their elementary properties. The grassmannian of lagrangian subspaces is given the structure of a Banach subvariety of the full grassmannian of complemented subspaces of a Banach space. We show that under the action of the symplectic group, orbits of lagrangian subspaces are diffeomorphic to Banach homogeneous spaces of the symplectic group.

Finally, in the special case of a Hilbert space, it is proved that the lagrangian grassmannian and, as a consequence, the linear symplectic group are contractible topological spaces. Examples are sketched which show that this result is false if either general Banach spaces or weak symplectic structures are allowed.

**1. Linear Symplectic Structures.** In the following **E** is a Banach space, equipped with a continuous skew-symmetric bilinear form $\omega : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{R}$. Define the "flatted" map $\omega_b : \mathbf{E} \rightarrow \mathbf{E}^*$ by setting $\omega_b(e) \cdot f = \omega(e, f)$. If $\omega_b$ is an isomorphism, then $\omega$ is (strongly) non-degenerate and the pair $(\mathbf{E}, \omega)$ is called a (strong) linear symplectic structure. If it happens that $\omega_b$ is merely injective, then $(\mathbf{E}, \omega)$ is a *weak* symplectic structure. Although equivalent in finite dimensions, the two notions differ for the general Banach space case; and, in fact, weak structures play a dominant rôle in J. Marsden's formulation [2] of infinite dimensional mechanics. Other possibilities lie between the extremes of weak or strong; e.g., Tromba [15] has defined an "almost symplectic" struc-

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