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LIE DERIVATIVES IN DIFFERENTIABLE SPACES

CHARLES D. MARSHALL

Dedicated to N. Aronszajn on the occasion of his seventieth birthday.

ABSTRACT. The Lie derivative $(\mathbf{L}_X t)p$ of a tensor field t at a point p on a differentiable space $\mathbf{S}$ may not be well defined. At each point $p \in S$ there is, however, a subspace $L_p\mathbf{S} \subseteq T_p\mathbf{S}$ such that $(\mathbf{L}_X t)p$ is well defined if and only if $X(p) \in L_p\mathbf{S}$. For any differentiable space, $L_p\mathbf{S} = T_p\mathbf{S}$ for every p in the complement of a nowhere dense subspace. In case $\mathbf{S}$ is either a coherent real-analytic space or a differentiable space of polyhedral type, then $L_p\mathbf{S} = T_p\mathbf{S}$ at every $p \in S$.

Introduction. On a differentiable manifold $\mathbf{M}$ the Lie derivative $\mathbf{L}_X t$ is well defined for any differentiable vector field X and any differentiable tensor field t. This need not be true, however, if M is more generally a differentiable space having singular points (and if the covariant rank of t is positive). The purpose of this note, then, is to characterize those vector fields X on a differentiable space such that $L_X t$ is well defined for every differentiable tensor field t. In fact, for each point p of a differentiable space $\mathbf{S}$ we identify a subspace $L_p\mathbf{S}$ of the tangent space $T_p\mathbf{S}$ such that $(\mathbf{L}_X t)p$ is well defined for every t if and only if $X(p) \in L_p\mathbf{S}$.

In § 1 we review some notions about differentiable spaces from a different point of view than that of [4]. In §2 we give examples showing that $\mathbf{L}_X t$ need not always be well defined, characterize $L_p\mathbf{S}$, and discuss the effects on $L_p\mathbf{S}$ of weakening the differentiable structure of $\mathbf{S}$. This characterization of $L_p\mathbf{S}$ together with a general result from §1 shows that the set of points p where $(\mathbf{L}_X t)p$ is possibly not well defined is always nowhere dense in $\mathbf{S}$. In §3 we apply these results to show that every C^∞ vector field on a coherent real analytic space gives well defined Lie derivatives. In §4 we explicitly calculate $L_p\mathbf{S}$ when $\mathbf{S}$ is locally diffeomorphic to polyhedral subsets of cartesian spaces. This calculation shows that every differentiable vector field on polyhedral spaces gives well defined Lie derivatives.

Throughout the paper we use the notion of smoothness category introduced by Palais in [5]. For the reader's convenience we have included a short appendix at the end of §1 recapitulating the definition and several results from [5]. Without mention to the contrary, "smooth" will mean $\mathscr{C}$-smooth, where $\mathscr{C}$ is some smoothness category. Finally, throughout the paper $\mathbf{R}^1$, $\mathbf{R}^n$ will denote the real cartesian spaces.

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