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THE C^*-ALGEBRA OF THE ELLIPTIC BOUNDARY PROBLEM

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0. **Introduction.** Let $\mathbf{R}_+^{n+1} = \{x = (x_0, \ldots, x_n): x_0 > 0\}$, and $\partial\mathbf{R}_+^{n+1} = \{x_0 = 0\}$. Consider unbounded differential operators L of $\mathfrak{H} = L^2(\mathbf{R}_+^{n+1})$ given by an expression $(a) = \sum_{|\alpha| \leqq N} a_\alpha D^\alpha$ over $\mathbf{R}_+^{n+1}$ and a set (b) of boundary expressions $(b_j) = \sum_{|\alpha| \leqq N_j} b_{j,\alpha} D^\alpha$, $N_j < N$, $j = 1, \ldots, m$. L is defined by (a), in $\operatorname{dom} L = \{u \in \mathfrak{H}_N$: $(b)u = 0\}$, with the L^2-Sobolev space $\mathfrak{H}_N = \mathfrak{H}_N(\mathbf{R}_+^{n+1})$. General assumptions: $a_\alpha^{(\beta)} \in CS(R_+^{n+1})$ $b_{j,\alpha}^{(\beta)} \in CS(\partial\mathbf{R}_+^{n+1})$, with the two C^*-function algebras over $\mathbf{R}_+^{n+1}$ and its boundary generated by $\lambda(x) = (1 + x^2)^{-1/2}$ and $s_j(x) = x_j\lambda(x)$, $j = 0, \ldots, n$, respectively.

Examples are the operators Δ_d and Δ_n, formed with the Laplace operator $(a) = \Delta$, and the Dirichlet and Neumann condition, $(b) = 1$, and $(b) = \partial/\partial x_0$, respectively. Δ_d and Δ_n are known to be negative self-adjoint operators of $\mathfrak{H}$, so that all operators of (0.1), below, are well defined bounded operators of $\mathfrak{H}$.

$$\text{(0.1)} \qquad \begin{aligned} &\Lambda_d = (1 - \Delta_d)^{-1/2}, \ \Lambda_n = (1 - \Delta_n)^{-1/2}, \ S_d = D_0\Lambda_d, \\ &S_n = D_0\Lambda_n, \ S_{j,d} = D_j\Lambda_d, \ S_{n,j} = D_j\Lambda_n, \ j = 1, \ldots, n. \end{aligned}$$

The C^*-algebras generated by (taking operator norm closure in $\mathfrak{L}(\mathfrak{H})$ of the finitely generated algebra of the operators) (0.1), (or (0.1) together with the multiplication operators $a(M): \mathfrak{H} \to \mathfrak{H}$, defined by $(a(M)u)(x) = a(x)u(x)$, $x \in \mathbf{R}_+^{n+1}$, for $a \in CS(\mathbf{R}_+^{n+1})$) will be denoted by $\mathfrak{A}^\#$ and $\mathfrak{A}$, respectively. We shall refer to $\mathfrak{A}$ as of *the C^*-algebra of the elliptic boundary problem* in the half space $\mathbf{R}_+^{n+1}$. We believe this distinctive notation justified, because the algebra $\mathfrak{A}$ proves to be of interest for a variety of reasons, listed below. First, c.f. **[10]**, $\mathfrak{A}$ contains (Fredholm) inverses L^{-1} of L generated by a general (Lopatinski—Shapiro type) variable coefficient boundary condition (b) and a suitable elliptic constant coefficient (a). Moreover we then even have $P_{L,\alpha} = D^\alpha L^{-1} \in \mathfrak{A}$, for all $|\alpha| \leqq N =$ order of L. Second, we shall make available good criteria for $A \in \mathfrak{A}$ to be Fredhom. Third, $\mathfrak{A}$ may be of interest as a type-1 C^*-algebra with a finite ideal chain

$$\text{(0.2)} \qquad \mathfrak{A} \supset \mathfrak{E} \supset \mathfrak{C},$$

where $\mathfrak{E}$ and $\mathfrak{C}$ denote the commutator ideal of $\mathfrak{A}$ and the compact ideal of $\mathfrak{H}$, respectively. In fact we get

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