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CONVERGENCE OF APPROXIMATION METHODS FOR EIGENVALUES OF COMPLETELY CONTINUOUS QUADRATIC FORMS

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DEDICATED TO N. ARONSZAJN

0. **Introduction.** In 1951 Aronszajn [5] outlined a general procedure for approximating the eigenvalues of a real quadratic form $\mathfrak{B}$ relative to a positive definite quadratic form $\mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are defined on a vector space V and $\mathfrak{B}$ is completely continuous with respect to $\mathfrak{A}$. This procedure encompassed all the then known variational approximation techniques, including the Rayleigh-Ritz, Weinstein, and Aronszajn methods. Briefly stated, the procedure is as follows:

The original eigenvalue problem is replaced by an auxiliary problem determined by a pair of forms $(\mathfrak{B}_0, \mathfrak{A}_0)$. The auxiliary problem is so chosen as to be explicitly solvable and such that its eigenvalues give approximations, however bad, to the corresponding eigenvalues of the original problem. These initial approximations are improved by introducing intermediate problems, whose eigenvalues give successively better (or at least no worse) approximations to those of the original problem. The auxiliary problem and the intermediate problems are chosen using the so-called monotony principles, and the eigenvalues of the intermediate problems are computed using those of the auxiliary problem and certain perturbation determinants (often called Weinstein-Aronszajn determinants). Depending on the method used, one gets either upper or lower approximations to the eigenvalues of the original problem. By using different methods, one can obtain both upper and lower approximations (and therefore a posteriori error estimates).

In [5] Aronszajn described how the above procedure could be applied in the case of four basic approximation methods, derived the corresponding Weinstein-Aronszajn determinants, and investigated the convergence of the approximating eigenvalues to the corresponding eigenvalues of the original problem. For three of the methods considered he was able to show the convergence under relatively mild and "natural" hypotheses. For the fourth method (Aronszajn's method) he was able to show it only under the additional hypothesis that $\mathfrak{A}_0$ be equivalent to $\mathfrak{A}$. He left open the questions of whether this hypothesis is necessary or whether conver-

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