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AN ANALOGUE OF BÄCKLUND'S THEOREM IN AFFINE GEOMETRY

SHIING-SHEN CHERN* AND CHUU-LIAN TERNG**

ABSTRACT. It is well-known that there is a correspondence between solutions of the Sine-Gordon equation (SGE)

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi$$

and the surfaces of constant curvature -1 in $\mathbf{R}^3$ (see below). The classical Bäcklund transformation of such surfaces furnishes a way to generate new solutions of the SGE from a given solution. This has received much attention in recent studies of the soliton solutions of the SGE, and the technique has been used successfully in the study of other non-linear evolution equations. In the first section of this paper we present a simple derivation of the classical Bäcklund theorem and its applications by using the method of moving frames.

Our main result concerns affine minimal surfaces. They arise as the solution of the variation problem for affine area. The corresponding Euler-Lagrange equation is a fourth order partial differential equation. In §2, we develop the basic properties of affine minimal surfaces. In §3 we study the transformation of affine surfaces by realizing them as the focal surfaces of a line congruence. The natural conditions that the congruence be a W-congruence and that the affine normals at corresponding points be parallel lead to the conclusion that both surfaces are affine minimal. This is the content of Theorem 4, the main result of our paper. As in the classical case, the Theorem leads to the construction of new affine minimal surfaces from a given one by the solution of a completely integrable system of first order partial differential equations.

1. **The classical Bäcklund theorem and its consequences.** Let M be a surface in $\mathbf{R}^3$. We choose a local field of orthonormal frames v_1, v_2, v_3 with origin X in $\mathbf{R}^3$ such that X is a point of M and the vectors v_1, v_2 are tangent to M at X. Let $\theta_1, \theta_2, \theta_3$ be the dual coframe of v_1, v_2, v_3. We can write

$$\begin{aligned} dx &= \sum_\alpha \theta_\alpha v_\alpha \\ dv_\alpha &= \sum_\beta \theta_{\alpha\beta} v_\beta, \end{aligned} \tag{1.1}$$

Here and throughout this paper we shall agree on the index ranges

$$1 \leqq i, j, k \leqq 2, \quad 1 \leqq \alpha, \beta, \gamma \leqq 3. \tag{1.2}$$

The structure equations of $\mathbf{R}^3$ are

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