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THE QUADRATIC AND QUARTIC CHARACTER OF CERTAIN QUADRATIC UNITS. II

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Let m be a square-free integer greater than 1, and let ϵ_m denote the fundamental unit of the real quadratic field $Q(\sqrt{m})$. If k is an integer not divisible by the odd prime p and the Legendre symbol (k/p) has the value 1, we define the symbol $(k/p)_4$ to be $+1$ or -1 according as k is or is not a fourth power modulo p. Now if $(m/p) = +1$ we can interpret ϵ_m as an integer modulo p and ask for the value of (ϵ_m/p). Because of the ambiguity in the choice of $\sqrt{m}$ taken modulo p we must make sure that (ϵ_m/p) is well defined. This is the case if ϵ_m has norm $+1$ (written $N(\epsilon_m) = 1$) or if $N(\epsilon_m) = -1$ and $p \equiv 1 \pmod 4$. Whenever $(\epsilon_m/p) = 1$ we can ask for the value of $(\epsilon_m/p)_4$. This latter symbol is well defined if $N(\epsilon_m) = +1$ or if $N(\epsilon_m) = -1$ and $p \equiv 1 \pmod 8$. The evaluations of these symbols are generally given in terms of representations of a power of p by certain positive-definite binary quadratic forms. This is convenient when considering applications to divisibility properties of recurrence sequences (see for example [14]).

An early result in this direction was proved by Barrucand and Cohn [1] who showed, using the arithmetic of $Q(\sqrt{-1}, \sqrt{2})$, that if $p \equiv 1 \pmod 8$ is prime, so that $p = c^2 + 8d^2$, then

$$(\epsilon_2/p) = (-1)^d.$$

This gives a criterion for the splitting of p in the non-abelian number field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{\epsilon_2})$.

Using similar methods, the present authors [17] have evaluated explicitly (ϵ_m/p) (when $N(\epsilon_m) = -1$) and $(\epsilon_m/p)_4$ (when $N(\epsilon_m) = +1$) for certain values of m, namely, those for which at least one of the imaginary bicyclic biquadratic fields

$$(1) \qquad Q(\sqrt{m}, \sqrt{-m}),\ Q(\sqrt{-m}, \sqrt{-2m}), \text{ or } Q(\sqrt{-2m}, \sqrt{m}),$$

has class number one (21 fields in all, see [6]). In this paper we extend these results to an infinite class of values of m. Our results and conjectures arise from those of the above fields (1) which have class number not divisible by 4.

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