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GROWTH OF DERIVATIVES AND THE MODULUS OF CONTINUITY OF ANALYTIC FUNCTIONS

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1. **Introduction.** Let G be a bounded complex domain and let $f(\xi)$ be analytic on G and continuous on $\overline{G}$. The modulus of continuity of $f(\xi)$ on $\overline{G}$ is a function $\omega(\delta, f, \overline{G})$ defined for $\delta \geqq 0$ by

$$\omega_f(\delta) = \omega(\delta, f, \overline{G}) \quad \sup_{\substack{\xi_1, \xi_2 \in \overline{G} \\ |\xi_1 - \xi_2| \leqq \delta}} |f(\xi_1) - f(\xi_2)|. \tag{1}$$

If

$$\omega_f(\delta) \leqq C\delta^\alpha,$$

for some $0 < \alpha \leqq 1$ and some constant $C > 0$, then $f(\xi)$ satisfies a Lipschitz condition of order α on $\overline{G}$.

If $G = D = \Delta(0, 1)$ is the open unit disk, a classical theorem of Hardy and Littlewood [1] shows that $f(\xi)$ satisfies a Lipschitz condition of order α on $\overline{D}$ if and only if

$$|f'(\xi)| \leqq C(1 - |\xi|)^{\alpha - 1}$$

for all $\xi \in D$. The positive constant C is independent of ξ. By conformal mapping, the Hardy-Littlewood theorem can be generalized to the case in which G is replaced by a bounded, simply connected domain G with analytic boundary. In particular, if

$$d(\xi, \partial G) = d_\xi = \inf_{z \in \partial G} |\xi - z|$$

denotes the distance from a point $\xi \in G$ to ∂G, then the following result holds [4].

THEOREM 1. *Let G be a bounded, simply connected domain with analytic boundary. A function $f(\xi)$ analytic on G and continuous on $\overline{G}$ satisfies a Lipschitz condition of order α on $\overline{G}$ if and only if*

$$|f'(\xi)| \leqq C\{d_\xi\}^{\alpha - 1},$$

for all $\xi \in G$.

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