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CAUCHY TRANSFORMS OF MEASURES, AND A CHARACTERIZATION OF SMOOTH PEAK INTERPOLATION SETS FOR THE BALL ALGEBRA

ALEXANDER NAGEL*

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n) \in \mathbf{C}^n$, let $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$, and let $|z| = \langle z, z \rangle^{1/2}$. Let $B_n = \{z \in \mathbf{C}^n | \ |z| < 1\}$ denote the unit ball in $\mathbf{C}^n$, and let $\partial B_n = \{z \in \mathbf{C}^n | \ |z| = 1\}$ denote its boundary. If $F(z)$ is holomorphic on B_n, we say that F belongs to $H^p(B_n)$, $0 < p < \infty$, if

$$\sup_{r<1} \int_{\partial B_n} |F(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

where $d\sigma$ is rotation invariant Lebesgue measure on ∂B_n. We say that $F \in H^\infty(B_n)$ if $\sup_{z \in B_n} |F(z)| < \infty$. If $F \in H^p(B_n)$ for $0 < p \leqq \infty$, then F has radial limits $F^*(\zeta)$ almost everywhere on ∂B_n with respect to $d\sigma$. Moreover, if $1 \leqq p < \infty$, $F(r\zeta)$ converges in L^p to $F^*(\zeta)$. (For a discussion of H^p theory in B_n, see for example Stein [6] or Stout [7].)

Let $d\mu$ be a finite Borel measure on ∂B_n. We shall denote by $C(\mu)$ the *Cauchy transform* of $d\mu$ which is given by

$$C(\mu)(z) = \int_{\partial B_n} [1 - \langle z, \zeta \rangle]^{-n} \, d\mu(\zeta).$$

$C(\mu)(z)$ is holomorphic on B_n, but in general it need not belong to $H^1(B_n)$, for example if $d\mu$ is a point mass.

The object of this paper is twofold. First we study $C(\mu)$ when $d\mu$ is "Lebesgue measure" on a smooth curve $\gamma \subset \partial B_n$. We show that if the tangent to the curve γ does not lie in the maximal complex subspace of the real tangent space to ∂B_n at each point, then $C(\mu)(z)$ does belong to $H^1(B_n)$, and in fact has better behavior depending on the smoothness of γ. (Note that when $n > 1$, it follows that $C(\mu)$ may belong to $H^1(B_n)$ even if $d\mu$ is singular with respect to the surface measure $d\sigma$ on ∂B_n.) Precise statements are given in Theorem 1.

A second object of this paper is to apply Theorem 1 to obtain a necessary condition for a compact set $K \subset \partial B_n$ to be a peak interpolation set for the ball algebra $A(B_n)$ of functions continuous on $\bar{B}_n$ and holomorphic on B_n. (For the definition of peak interpolation set, see section

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