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DIFFERENTIABLE POINTS OF THE GENERALIZED CANTOR FUNCTION

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ABSTRACT. The generalized Cantor function Θ_γ has a derivative equal to $1/(1 - \gamma)$ at almost every point in the set C_γ. This was established by Darst [1] who then posed the problem of characterizing those points which are not differentiable. The differentiability of points in C_γ is determined by the spacing of the 0's and 2's in a ternary-like expansion. Points that are interval endpoints have one-sided derivatives from both sides.

1. **Introduction.** To describe a generalized Cantor set, denoted by C_γ, and the corresponding Cantor function Θ_γ, first choose a number γ satisfying $0 < \gamma < 1$. The usual Cantor set is obtained when $\gamma = 1$. The set C_γ is obtained in the same manner as the standard Cantor set by deleting a sequence $\{(a_i, b_i)\}_{i=1}^\infty$ of pairwise disjoint segments from the interior of the unit interval. In general, the k-th step consists of removing an open interval of length $\gamma/3^k$ from the middle of each of the 2^{k-1} closed intervals, thereby leaving 2^k closed intervals of equal length. This length is in fact equal to $(1 - \gamma_k)/2^k$, where $\gamma_k = \gamma[1 - (2/3)^k]$. The process continues, and C_γ is defined to be the set of points in $[0, 1]$ which fail to be removed. The measure of C_γ is positive and equals $1 - \gamma$. The corresponding Cantor function is defined analogously to the standard Cantor function. The function Θ_γ is a nonnegative, nondecreasing continuous function. In addition, Darst established that $\Theta_\gamma'(x) = 1/(1 - \gamma)$ for almost all x in C_γ. Characterizing the set of points in $[0, 1]$ at which Θ_γ is not differentiable is the problem this paper concerns itself with.

2. **Derivatives at Endpoints.** In establishing $\Theta_\gamma'(x) = 1/(1 - \gamma)$ for almost all x in C_γ, Darst showed that

$$\left| \frac{\Theta_\gamma(y) - \Theta_\gamma(x)}{y - x} \right| \leqq \frac{1}{1 - \gamma}$$

for all x, y in $[0, 1]$ with $x \neq y$. Our first result is that all right (left) hand interval endpoints have derivatives from the right (left) which equal $1/(1 - \gamma)$. A geometric approach will be used and a sketch of the proof given. To proceed, let x be an arbitrary right endpoint, where the length of the removed interval is $\gamma/3^k$ and k is some positive integer. For each integer $n > k$, let $J_n = (u_n, v_n)$ be the removed in-

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