A NOTE ON EXCHANGEABLE SEQUENCES OF EVENTS

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ABSTRACT. Bruno de Finetti's (1931) representation for the law of an exchangeable sequence of 0's and 1's is exhibited as an invariant limit in the ergodic theorem for a transformation first defined by L. K. Arnold (1968) and studied by Hajian, Ito, and Kakutani (1972) in the context of σ-finite invariant measures. A known result on almost-sure convergence of normalized sums for such a sequence emerges as a corollary.

0. Background. A random sequence \( \{X_k(\omega)\}_{k=1}^{\infty} \) on a probability space \((\Omega, \mathcal{F}, \mu)\) is said to be exchangeable if for any permutation \(\sigma\) of a finite set \(\{k_1, \ldots, k_n\}\) of indices and for any \(A_1, \ldots, A_n \in \mathcal{F}\) measurable events in \(\Omega\),

\[
\mu(\{\omega \in \Omega : X_{\sigma(k_1)}(\omega) \in A_1, \ldots, X_{\sigma(k_n)}(\omega) \in A_n\}) = \mu(X_{k_1} \in A_1, \ldots, X_{k_n} \in A_n),
\]

where we suppress explicit mention of \(\omega\) on the right-hand side. A sequence of measurable events \(\{A_k\}_{k=1}^{\infty}\) is called exchangeable whenever the sequence \(\{I_{A_k}(\cdot)\}\) of its indicator functions is.

The simplest example of exchangeable events is that of an independent sequence \(\{A_k\}_{k=1}^{\infty}\) with all \(\mu(A_k)\) equal. This case occurs when the indicator \(I_{A_k}(\omega) = \omega_k\) is the \(k^{th}\) coordinate of a point \(\omega \in \{0, 1\}^\infty \equiv \Omega\), where \(\mathcal{F}\) is the product \(\sigma\)-algebra and \(\mu\) the infinite-product probability measure on \(\{0, 1\}^\infty\) assigning probability \(\mu(A_1) = \mu(A_k)\) to \(\{1\} \times \{0, 1\}^\infty\).

Given any sequence \(\{A_k\}_{k=1}^{\infty}\) of measurable events in \(\Omega\), we can identify the measure spaces \((\Omega, \mathcal{F})\) and \(\{0, 1\}^\infty\) via \(\omega \mapsto \{I_{A_k}(\omega)\}_{k=1}^{\infty}\).

So from now on we take \(\Omega = \{0, 1\}^\infty\) with product \(\sigma\)-algebra \(\mathcal{F}\), so that \(\mu\) is the probability law of the random sequence \(\omega = (\omega_1, \omega_2, \ldots) \in \{0, 1\}^\infty\). We assume that \(\{X_k(\omega)\}_{k=1}^{\infty} \equiv \{I_{A_k}(\omega)\}_{k=1}^{\infty} = \{\omega_k\}_{k=1}^{\infty}\) is exchangeable, and call the measure \(\mu\) exchangeable as well.

A celebrated theorem of Bruno de Finetti [2] says that the most general exchangeable measure \(\mu\) on \(\{0, 1\}^\infty\) is a mixture of infinite-product measures. There are many ways to prove this, including a particularly elementary combinatorial one due to Feller [3, p. 228]. In this paper we give a proof intended to shed immediate light on a further analogy between exchangeable and independent sequences: