ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

PERIODIC SOLUTIONS OF AUTONOMOUS DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONAL SPACES

WILLIAM GRASMAN

1. **Introduction.** The Poincaré-Bendixson theorem supplies a powerful technique for finding periodic orbits of dynamical systems in the plane. As is well known, the theorem does not apply in higher dimensional spaces since it depends on the Jordan curve theorem.

Even though the Poincaré-Bendixson theorem does not apply to higher dimensional systems, some higher-dimensional analogies to results for the plane are hard to disbelieve. For example, suppose a dynamical system in R^n, $n \geqq 3$, leaves the n-disk D^n invariant in the positive direction and has a single rest point which has an unstable manifold of dimension at least two. It seems very likely, at first, that such a system must have a periodic orbit.

Such results are not valid. The method which Schweitzer [11] used to disprove the Seifert conjecture also applies to this and similar situations. Schweitzer's method will be outlined in § 3.

There must be additional hypotheses on the vector field in order to insure the existence of periodic orbits. Rauch [9] and others have used a technique of finding a positively invariant solid torus and showing that a particular flow "twists" around the torus. This "twisting" hypothesis allows one to construct a "first return" map and apply the Brouwer fixed-point theorem to find a periodic orbit.

The process of finding a positively invariant solid torus can be very difficult. For equations that model ecological systems, in particular, it is much more feasible to find a positively invariant disk. In § 2 we state and prove a theorem about systems which have positively invariant disks and satisfy a "twisting" condition and in § 4, we apply the theorem of § 2 to an ecological model of three competing species.

Throughout this paper, R is the set of real numbers, R^n is Euclidean n-space, $C^k(M, N)$ is the set of k-times continuously differentiable maps from M to N, and $C^k(M) = C^k(M, R)$.

I would like to thank Z. Nitecki and J. A. Yorke for a conversation at the 1976 NSF-CBMS conference on topological methods in differential equations, which clarified Schweitzer's method for me. Professor Nitecki must also be credited with the first remark following the statement of the theorem in § 2. I would also like to thank my advisor, Paul Waltman, for suggesting this problem and for his encouragement and patience.