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EXTREMAL PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. Denote by H the subclass of close-to-convex functions $f(z)$ for which there exists a starlike function $g(z)$ satisfying $\mathrm{Re}\{[zf'(z)]'/g'(z)\} > 0$ $(|z| < 1)$. We find distortion theorems, coefficient bounds, and the closed convex hull of H. We also give a necessary intrinsic condition for a function to be in H.

1. **Introduction.** Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the disk $|z| < 1$. A function $f(z) \in S$ is said to be *starlike* if $\mathrm{Re}\{zf'(z)/f(z)\} > 0(|z| < 1)$, is said to be *convex* if $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0(|z| < 1)$, and is said to be *close-to-convex* if there exists a starlike function $g(z)$ such that $\mathrm{Re}\{zf'(z)/g(z)\} > 0(|z| < 1)$. These classes are denoted respectively by S^*, K, and C.

We denote by H the class of functions of the form (1) for which there exists a function $g(z) \in S^*$ such that

$$\mathrm{Re}\left\{\frac{[zf'(z)]'}{g'(z)}\right\} > 0(|z| < 1). \tag{2}$$

In [5] Sakaguchi shows for $g(z) \in S^*$ that $\mathrm{Re}\{[zf'(z)]'/g'(z)\} > 0$ implies $\mathrm{Re}\{zf'(z)/g(z)\} > 0$. Thus $H \subset C$. Moreover if $f(z) \in K$, then $\mathrm{Re}\{[zf'(z)]'/f'(z)\} = \mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$. Hence we may take $g(z) = f(z)$ in (2) to show that $f(z)$ is also in H. Thus $K \subset H$.

It is well known that $K \subset S^* \subset C$. Since we also have the inclusion relations $K \subset H \subset C$, it is of interest to inquire as to the relationship between S^* and H. In the next section, we shall show that S^* is neither contained in nor contains H.

Note that the result of Sakaguchi yields a quick proof that $K \subset S^*$, for $\mathrm{Re}\{[zf'(z)]'/f'(z)\} > 0$ implies $\mathrm{Re}\{zf'(z)/f(z)\} > 0$.

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