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ASYMPTOTIC APPROXIMATIONS TO THE SOLUTION OF THE HEAT EQUATION

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ABSTRACT. In one-dimensional problems of diffusion or heat conduction where discontinuities or steep gradients occur in the initial or boundary conditions a singular perturbation analysis can give accurate estimates of the solution when numerical methods prove inefficient or inadequate. In fact, the discontinuities can be exploited in the singular perturbation analysis to obtain an asymptotic series representation of the solution.

Several different problems of increasing complexity can be explicitly solved when the boundary and initial data are given in piecewise polynomial form: (a) infinite region or pure initial value problem, (b) semi-infinite region, and (c) finite region.

The approximate methods also apply to the case when no discontinuities occur in the prescribed data or its derivatives.

1. **Introduction.** It is frequently stated in regard to heat conduction and diffusion problems that "discontinuities are immediately damped out". In some problems arising in engineering, however, this view is too over-simplified to be realistic because the damping out process itself is the heart of the problem. Typically, such problems exhibit a behavior usually referred to as "very steep gradients" which present severe difficulties to attempted solution by numerical techniques; viz., very small mesh sizes and excessive roundoff errors. We shall examine in this paper how such problems are most suitably handled by a singular perturbation analysis. The analysis will show how very accurate estimates of the solution can be obtained with just a few easily calculated terms.

We first wish to give an example of the type of problem we have in mind, namely, heat conduction in rifle barrels. We assume circular symmetry and independence of the axial coordinate. The phenomenon is then described by the heat conduction equation in radial coordinates:

$$u_t = a[u_{rr} + (1/r)u_r].$$

The radial coordinate varies from r_0 = interior radius of the barrel to r_1 = exterior radius. The initial and boundary (radiation) conditions are

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