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CHARACTERS OF THE WEYL GROUP OF $SU(n)$ ON ZERO WEIGHT SPACES AND CENTRALIZERS OF PERMUTATION REPRESENTATIONS

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1. **Introduction.** If G is a compact simple Lie group with maximal abelian subgroup T and normalizer $N(T)$, then $W = N(T)/T$ is a finite group called the *Weyl group* of G. If $\mathcal{G}$ is the Lie algebra of G with $\mathcal{T}$ the Cartan subalgebra corresponding to T, then the adjoint action of G on $\mathcal{G}$ has the property that $\mathcal{T} = \{x \in \mathcal{G} : t \cdot x = x \text{ for all } t \in T\}$. Thus $\mathcal{T}$ is naturally a W-module and it is well-known that W acts on $\mathcal{T}$ as a group generated by reflections. A generalization of this situation is the following. Let M be a complex G-module and let $M_0 = \{x \in M : t \cdot x = x \text{ for all } t \in T\}$, the *zero-weight space* of M. Then M_0 is naturally a W-module. It is the purpose of this paper to characterize the W-module structure of M_0 in case $G = SU(V)$ (where V is n-dimensional unitary space) and M is a finite dimensional simple G-module.

REMARK. The structure of M_0 as a W-module is closely related to the structure of H, the graded G-module of G-harmonic polynomials over $\mathcal{G}$. For example, the multiplicity of M in H is exactly $k = \dim(M_0)$. Furthermore, if $m_1, \cdots, m_k$ are the homogeneous degrees of H in which M occurs (the *generalized exponents* of M), then the eigenvalues in M_0 of a Coxeter-Killing element in W are just $\exp(2\pi i/m_j)$ $(j = 1, \cdots, k)$. See Kostant's paper [3] for a definition of the G-harmonic polynomials and more details.

Our results for $G = SU(V)$ depend heavily on the classical correspondence between the irreducible representations of $SU(V)$ and those of the symmetric groups S_m as m ranges over all positive integers. This correspondence is due to the fact that the linear span of the action of S_m on $\otimes^m V$ is the full centralizer of the action of $SU(V)$ on $\otimes^m V$. In § 2, we will summarize this correspondence using a more general result about centralizing group representations. In § 3 we will prove a sharpened version of this result for permutation representations of finite groups. Finally, in § 4 we will obtain a formula for the character of W on M_0 related to Littlewood's plethysm of S-functions.

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