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POSITIVE DEFINITE FUNCTIONS AND GENERALIZATIONS, AN HISTORICAL SURVEY

JAMES STEWART

1. **Introduction.** A complex-valued function f of a real variable is said to be *positive definite* (abbreviated as p.d.) if the inequality

(1.1) $$\sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqq 0$$

holds for every choice of $x_1, \cdots, x_n \in R$(the real numbers) and $\xi_1, \cdots, \xi_n \in C$(the complex numbers). In other words,the matrix

(1.2) $$[f(x_i - x_j)]_{i,j=1}^{n}$$

is positive definite (strictly speaking we should say positive semi-definite or non-negative definite) for all n, no matter how the x_i's are chosen. A synonym for positive definite function is *function of positive type*.

For example, the function $f(x) = \cos x$ is p.d. because

$$\sum_{i,j=1}^{n} \cos(x_i - x_j)\xi_i\bar{\xi}_j = \sum_{i,j=1}^{n} (\cos x_i \cos x_j + \sin x_i \sin x_j)\xi_i\bar{\xi}_j = \left| \sum_{i=1}^{n} \xi_i \cos x_i \right|^2 + \left| \sum_{i=1}^{n} \xi_i \sin x_i \right|^2 \geqq 0.$$

Likewise it is easily verified directly that $e^{i\lambda x}$ is p.d. for real λ, but it is not so straightforward to see that such functions as $e^{-|x|}$, e^{-x^2}, and $(1 + x^2)^{-1}$ are p.d. These and other examples are discussed in § 3.

Positive definite functions and their various analogues and generalizations have arisen in diverse parts of mathematics since the beginning of this century. They occur naturally in Fourier analysis, probability theory, operator theory, complex function-theory, moment problems, integral equations, boundary-value problems for partial differential equations, embedding problems, information theory, and other areas. Their history constitutes a good illustration of the words of Hobson [**51**, p. 290]: "Not only are special results, obtained independently of one another, frequently seen to be really included in

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