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SOME LOGICAL PROBLEMS CONCERNING FREE AND FREE PRODUCT GROUPS*

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The problems discussed in this paper are all concerned with attempts to understand the first order theories of free groups and the closely related groups obtained by the free product construction. Some of these problems are as old as model theory itself, and appear in the earliest writings on that subject by Tarski [31, 33] and Malc'ev [19]. All of these problems appear to be very difficult, and it is only in the last several years that substantial progress has been made toward their solutions. Already these results have uncovered interesting new areas of model theory, such as projective model theory [26, 27, 28] and new techniques in combinatorial group theory, and further work on these problems promises substantial new insights into both algebra and model theory.

The *first order* (or *elementary*) *language* L of group theory can be described as follows: the symbols of L include the constant 1, the variables $x_1, x_2, \cdots$, which range over group elements, the symbols $\cdot$ and $^{-1}$ for the two group operations, the equality symbol $=$, and the logical symbols $\sim$ (negation), & (conjuction), $\vee$ (disjunction), $\forall$ (for all group elements $\cdots$), and $\exists$ (there is a group elements such that $\cdots$). The *atomic formulas* of L include all formulas of form $W = W'$ where W and W' are products of the variables, the constant, and their inverses. The class C of *well-formed formulas* of L is the least collection of formulas such that C contains all atomic formulas, and if α and β are in C and v is any variable, then $(\sim\alpha)$, $(\alpha \,\&\, \beta)$, $(\alpha \vee \beta)$, $\forall v\alpha$ and $\exists v\alpha$ are all in C. The sentences of L are the well-formed formulas α such that if v is any variable which occurs in α, then v only occurs in well-formed subformulas of α of form $\forall v\beta$ or $\exists v\beta$. Every sentence of L is logically equivalent to a sentence of form $Q_1x_1 \cdots Q_nx_nM$, where M is a Boolean combination of atomic formulas and each Q_i is either $\forall$ or $\exists$. In particular, if all Q_i's are $\forall$, then the formula is termed *universal.*

Two groups are *elementarily equivalent* if they satisfy precisely the same sentences of L. If G is any group, let L_G be the language ob-

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