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RIEMANN'S FUNCTIONAL EQUATION

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It is well known that $\zeta(s)$, the Riemann Zeta function, satisfies the functional equation

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1}$$

In 1921, it was shown by Hamburger (See [2]) that $\zeta(s)$, as a member of a wide class of ordinary Dirichlet series, could be characterized by equation (1). Hamburger considered, in fact, a more general problem, the solution of the functional equation

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)f(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)g(1-s), \tag{2}$$

where "solution" here (and for the balance of the paper) refers to a pair of Dirichlet series, $(f(s), g(s))$ satisfying equation (2). Hamburger imposed conditions on $f(s)$ and $g(s)$ which, together with the fact that they satisfy equation (2), necessitated that they satisfy $f(s) = g(s) = c\zeta(s)$.

Subsequently, other researchers have found different sets of conditions, which, when imposed on the solutions of (2), have again led to the same conclusion that $f(s) = g(s) = c\zeta(s)$. (See [1] and [5].) For example, a corollary to a theorem in [5] is the

THEOREM 1. *Let* $f(s) = \sum_{j=1}^{\infty} a_j\mu_j^{-s} = \prod_{j=1}^{\infty}(1-\pi_j^{-s})^{-1}$, $1 < \pi_1 \leqq \pi_2 \leqq \cdots, \pi_j \to \infty$, *be a general Dirichlet series, with an Euler product representation, which converges for* $\mathrm{Re}(s) > 1$. *Suppose that* $f(s) = E(s)(s-1)^{-1}$, *where* $E(s)$ *is an entire function of finite order such that* $E(1) = 1$ *and* $E(0) = 1/2$. *Further, let* $g(s) = \sum_{k=1}^{\infty} b_k\nu_k^{-s}$, $1 \leqq \nu_1 < \nu_2 < \cdots, \nu_k \to \infty$, *be a general Dirichlet series which converges absolutely for* $\mathrm{Re}(s) \geqq 2$. *Then, if* $f(s)$ *and* $g(s)$ *are related by equation* (2), *it follows that* $f(s) = g(s) = \zeta(s)$.

We mention that an essential difference between the hypotheses of Hamburger's theorem and the hypothesis of Theorem 1, is that Hamburger assumed that $\mu_j \in Z^+, j = 1, 2, \cdots$, whereas in Theorem 1, $f(s)$ may be a general Dirichlet series, albeit with an Euler product representation.

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