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ON A CRANK-NICOLSON SCHEME FOR NONLINEAR PARABOLIC EQUATIONS

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1. **Introduction.** In this paper we consider a Crank-Nicolson type scheme for the problem:

$$u_t = f(t, x, u, u_x, u_{xx}) \text{ in } (0, T] \times (a, b) \tag{1.1}$$

$$u(0, x) = \varphi(x), u(t, a) = \varphi_0(t), \text{ and } u(t, b) = \varphi_1(t) \tag{1.2}$$

where $\varphi(a) = \varphi_0(0)$ and $\varphi(b) = \varphi_1(0)$.

In [11], the author was able to obtain a convergence theorem for a set of finite difference analogues of (1.1), (1.2) with $(0, T] \times (a, b)$ replaced by $[0, T] \times (a, b)$. For the Crank-Nicolson type scheme included among the methods in [11] a $O(\Delta t + h^2)$ convergence result was obtained. No method was given for solving the nonlinear system of difference equations.

For the Crank-Nicolson type scheme presented here, three improvements are possible. We obtain $O((\Delta t)^2 + h^2)$ convergence, we give a convergent iterative scheme for solving the nonlinear system of difference equations, and we obtain our results without assuming that the solution of (1.1) has continuous derivatives at $t = 0$.

Consideration of this iterative procedure yields an existence and uniqueness theorem for the solution of the nonlinear system of difference equations. This existence and uniqueness theorem is a slight improvement over the analogous result in [11], in that we obtain it by requiring that $f(t, x, z, p, r)$ satisfies certain Lipschitz conditions with respect to z, p, and r whereas in [11], we assumed f had continuous partial derivatives with respect to z, p, and r.

2. **Notation and Preliminary Results.** Let

$$h = \frac{b-a}{n+1} \text{ and } \Delta t = T/m \tag{2.1}$$

where n and m are positive integers. Also let $x_i = a + ih$ for $i = 0, 1, \cdots, n + 1$ and $t_j = j\,\Delta t$ for $j = 0, 1, \cdots, m$.

For the remainder of the paper, we will suppose 2.1 defines a mesh on $[0, T] \times [a, b]$, and if $v(t, x)$ is any function defined on this mesh we denote $v(t_j, x_i)$ by $v_{i,j}$. For any such mesh function, we let

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