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ON THE INTERCHANGE OF ORDER IN REPEATED LIMITS

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This subject, one of the most fundamental in analysis, is here dealt with in an abstract setting so as to be applicable to as wide a variety of situations as possible. Even restricted to real-valued integer-indexed sequences the treatment offers some advantages over that to be found in the textbooks. Proofs, being both standard and easy, are omitted.

The framework at the outset will be a set X equipped with a non-negative real-valued function of two variables satisfying

$$\rho(y, x) = \rho(x, y)$$
$$\rho(x, z) \leqq \rho(x, y) + \rho(y, z) \quad .$$

$\rho(x, x) = 0$ will follow of itself for all x for which $\rho(x, X)$ is not bounded away from zero; $\rho(x, y) = 0$ for $x \neq y$ can be allowed.

Convergence on X will be introduced via the following notion [7]: by a *(generalized) sequence* on X I shall mean a triple consisting of an indexing set M, a filter base (i.e., a collection directed downward by inclusion) $\mathcal{F}$ of its nonvoid subsets, and a function x on M to X. $(M, \mathcal{F}, x)$ is called *equivalent to* $(N, \mathcal{G}, y)$ if

$$\inf_{A \in \mathcal{F}, B \in \mathcal{G}} \sup_{\mu \in A, \nu \in B} \rho(x(\mu), y(\nu)) = 0.$$

Being symmetric and transitive, this is an equivalence relation on its domain whose elements are called *Cauchy sequences*. A sequence equivalent to an element (considered as the sequence injecting that element into X) is said to *converge* to it; if the element is unique I call it the *limit* of the sequence, $\lim_{\mathcal{F}} x(\mu)$. Using convergence in the non-negative reals, the convergence of a sequence to an element can be formulated as $\lim_{\mathcal{F}} \rho(x(\mu), x) = 0$; more generally, the equivalence of $(M, \mathcal{F}, x)$ with $(N, \mathcal{G}, y)$ as $\lim_{\mathcal{F} \times \mathcal{G}} \rho(x)\mu), y(\nu)) = 0$, where $\mathcal{F} \times \mathcal{G}$ is the filter base on $M \times N$ of products $\{A \times B : A \in \mathcal{F}, B \in \mathcal{G}\}$.

Let now M and N be sets equipped with the respective filter bases $\mathcal{F}$ and $\mathcal{G}$, and let x be a function on $M \times N$ to X. By fixing a value of $\nu \in N$ I may regard x as a function on M whose limit, $\lim_{\mathcal{F}} x(\mu, \nu)$, if it exists, is a function on N whose limit, if it in turn exists, is called the *repeated limit* $\lim_{\mathcal{G}} \lim_{\mathcal{F}} x(\mu, \nu)$.

Another approach to this quantity is via the filter base $\mathcal{G}/\mathcal{F}$ on $M \times N$ which consists of the subsets $U_{\nu \in B} A_\nu \times \{\nu\}$ where the A_ν and B are chosen in $\mathcal{F}$ and $\mathcal{G}$ respectively in all possible ways. Thus $x =$

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