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A FORMAL NUMBER-TERMED NUMBER SYSTEM BASED ON RECURSION

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ABSTRACT. This is a formal number-termed axiomatization of number theory based on recursion without presupposing or developing general functional concepts or any part of set theory, together with an initial development to indicate its adequacy. The formal inferential system in which it is framed is that of A. P. Morse. The axioms, phrased in the primitive terms '0' and 'II xy $\underline{u}'$ xy m n' and the defined term 'scsr n', are essentially

(1) II $xy\,\underline{u}'\,xy\,m\,0 = m$,
(2) II $xy\,\underline{u}'xy\,m$ scsr $n = \underline{u}'n$ II $xy\,\underline{u}'xy\,mn$,
(3) induction.

Introduction. For full-blown (natural) number theory we require the capability of making recursive definitions. We need to define exponentiation and more generally arbitrary finite summation and multiplication, to introduce the Euler φ function, to state and prove the unique prime power factorization theorem, to justify general commutativity and associativity, to deal with permutations, combinations, partitions, etc.

In order to explore number theory it is usual to presuppose enough set theory to construct ordered pairs, relations and functions and then develop the apparatus needed to make recursive definitions. A modest set theory will suffice, even one whose sets are constrained to be finite [3; Ch. 1, Sec. 6]. However, there are several ways [5, pp. 81–85; 3, Ch. 1, Sec. 6; 7; 2; 8] in which number theory can be axiomatized without presupposing a set-theoretic setting. In each of these the early development, culminating in a point where recursive definitions may be made, is somewhat tortuous. In the system presented here we formally axiomatize recursion directly so that an immediate assault on recursive problems is possible. This is done without either general functional concepts or any part of set theory, in such a way that all terms denote only numbers or meaninglessness. Although we do not have the full facility of the set-theoretic approach with its capacity for term denotation of finite sets and sequences as well as

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