ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 4, Fall 1974

APPLICATIONS AND PROOF OF A UNIQUENESS THEOREM FOR LINEAR INVARIANT FAMILIES OF FINITE ORDER

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Let S be the set of all univalent analytic functions in the open unit disc of the form $f(z) = z + a_2 z^2 + \cdots$. One of the most important results concerning such normalized univalent functions, which has not only been extensively used to solve many extremal problems in S, but also to demonstrate the essential uniqueness of the solution, is the well known fact that

$$|a_2| \leqq 2, \tag{0.1}$$

with equality only for rotations of the Koebe function, $k(z) = z/(1 - z)^2$.

In this paper we continue our theme [1], [2], [3], [4], that many of the classical results for the family S find not only a more general but also a more meaningful setting in the context of linear invariant families of locally univalent functions of finite order (defined in § 1). The impetus for such an investigation is to be found in Ch. Pommerenke's fundamental papers [13], [14] which form the beginning of this subject.

If $f(z) \in S$, then equality holds in (0.1) only for rotations of the Koebe function, while if $f(z) \in M$, a linear invariant family of finite order α, then equality can hold in

$$|a_2| \leqq \alpha \tag{0.2}$$

for functions other than rotations of the generalized Koebe function. Thus we do not always have unique solutions to the extremal problem $\max\{|a_2| : f \in M\}$. In § 2, we show that the extremal problem $\max\{r : \text{each } f \text{ in } U_\alpha \text{ maps } |z| < r \text{ univalently onto a convex domain}\}$ does not have a unique solution. Theorem 1 gives an elementary proof that there is a unique solution to the radius of convexity problem in S.

Since so many of the uniqueness results for S have classically been derived from the uniqueness properties of inequality (0.1), and in light of the existence of nonunique solutions to extremal problems in U_α, it is a reasonable question to ask if uniqueness results are possible for the families U_α. We therefore prove Theorem 2, which is a uniqueness result for the fundamental distortion theorem of U_α. Theorem 2 is applied in § 4 to show that the generalized Koebe function is the

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Received by the editors December 31, 1971 and, in revised form May 9, 1972.
AMS 1970 *subject classifications.* Primary 30A36; Secondary 30A32, 30A26.