ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 3, Summer 1974

STOCHASTICALLY PERTURBED DYNAMICAL SYSTEMS

WENDELL H. FLEMING [1,2]

1. **Introduction.** A variety of natural phenomena have been modeled by stochastic differential equations containing one or more small parameters. Such models allow one to study the effects of random perturbations of small magnitude. When the parameters are set equal to 0, the stochastic effects disappear from the problem; the resulting ordinary differential equations describe the evolution of the unperturbed dynamical system. Examples can be found in the study of wave propagation in random media [17], of noise in electronic systems [19] or in a control system [6]. Yet another class of examples, which we mention in § 6, arises from population genetics models.

For simplicity, we consider only one small positive parameter ϵ. The state space of the system being modeled is taken as n-dimensional R^n. The state process $\xi^\epsilon = (\xi_1^\epsilon, \cdots, \xi_n^\epsilon)$ is assumed to be a Markov diffusion, with generator $\mathcal{L}^\epsilon$ the partial differential operator

$$\mathcal{L}^\epsilon \psi = \epsilon \operatorname{tr} \alpha \psi_{xx} + \psi_x \cdot f. \tag{1.1}$$

Here ψ denotes any function of class C^2 (continuous second order partial derivatives); the functions α, f are respectively matrix and vector valued:

$$\alpha(x) = (\alpha_{ij}(x)), \quad i,j = 1, \cdots, n,$$
$$f(x) = (f_1(x), \cdots, f_n(x)).$$

The vector $\psi_x = (\psi_{x_1}, \cdots, \psi_{x_n})$ is the gradient of ψ, and

$$\operatorname{tr} \alpha \Psi_{xx} = \sum_{i,j=1}^{n} \alpha_{ij} \Psi_{x_i x_j}.$$

In fact, we shall take ξ^ϵ as the solution of a stochastic differential equation of Ito type (§ 2).

A variety of questions can be asked about the perturbed process ξ^ϵ. Which of these has a reasonable answer in a particular instance

[1]This research was supported in part by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 71-2078, and in part by the National Science Foundation under Grant No. GP 15132.

[2]Presented at the Conference on Stochastic Differential Equations, University of Alberta, July 1972.

Received by the Editors June 22, 1973.