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# EXTRAPOLATION TO THE LIMIT BY USING CONTINUED FRACTION INTERPOLATION

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1. **The extrapolation problem.** Assume that a convergent sequence $\{a_0, a_1, a_2, \cdots\}$ of real numbers is given with $A$ as limit. In order to find the limit $A$ numerically one can form a new sequence $\{b_i\}$, which has also $A$ as limit and whose convergence is faster. One way to perform the determination of $\{b_i\}$ is to use extrapolation methods.

Let $\{x_0, x_1, \cdots\}$ be a convergent sequence of points with $z$ as limit. The essential idea in extrapolation is to define a sequence of interpolating functions $\{y_0(x), y_1(x), \cdots\}$ such that $y_n(x_i) = a_i$ for $i = 0, 1, \cdots n$ and $n = 0, 1, 2, \cdots$. The elements $b_i$ can be defined as follows $b_i = \lim_{x \to z} y_i(x)$ for $i = 0, 1, 2, \cdots$, if these limits exist and are finite. The points $x_i$ are called interpolation points and $z$ is called the extrapolation point.

Let $R(\ell, m)$ be the class of ordinary rational functions $r_{\ell,m} = p/q$ where the degree of $p$ is at most $\ell$ and the degree of $q$ at most $m$. Under certain conditions it is possible to construct a set of rational functions $r_{\ell,m}$ for $\ell, m = 0, 1, 2, \cdots$, satisfying $r_{\ell,m}(x_i) = a_i$ for $i = 0, 1, \cdots, \ell + m$. This set of functions can be arranged in a table as follows

| | | | | |
|---|---|---|---|---|
| $r_{0,0}$ | $r_{0,1}$ | $r_{0,2}$ | $r_{0,3}$ | — |
| $r_{1,0}$ | $r_{1,1}$ | $r_{1,2}$ | $r_{1,3}$ | — |
| $r_{2,0}$ | $r_{2,1}$ | $r_{2,2}$ | $r_{2,3}$ | — |
| — | — | — | — | — |

In the method of Neville (polynomial extrapolation) the first column is constructed. In the method of Bulirsch and Stoer (rational extrapolation) the "staircase" $r_{0,0}, r_{1,0}, r_{1,1}, r_{2,1}, \cdots$ is constructed. In both methods $z = 0$ is used as extrapolation point and this makes the calculation of $b_{\ell+m} = r_{\ell,m}(z)$ very simple.

The elements $r_{0,0}, r_{1,1}, r_{2,2}, \cdots$ can be found by using Thiele's method for continued fraction interpolation. If $z = \infty$ is taken as extrapolation point then the values of $b_i$ can be computed by using a method similar to the $\epsilon$-algorithm (see [1], p. 186 and [2]).

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