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SEQUENCES OF IRRATIONAL FRACTION APPROXIMANTS TO SOME HYPERGEOMETRIC FUNCTIONS

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I. **Introduction.** Let the Gauss continued fraction (C.F.) for $R(x) = \ln((x+1)/(x-1))$, with x real and $|x| > 1$, have convergents $L_s = N_s/D_s$, where for example, $N_0 = 0$, $D_0 = 1$; $N_1 = 2$, $D_1 = x$; $N_2 = 3x$, $D_2 = (3x^2 - 1)/2$, etc. Also let $u_s = D_s D_{s+2} - D^2_{s+1}$, $v_s = D_s N_{s+2} + D_{s+2}N_s - 2D_{s+1}N_{s+1}$, $w_s = N_s N_{s+2} - N^2_{s+1}$. Then I have shown [1] that if

$$R_s = \{(s+2)(s+1)v_s + 2\{(2s+3)^2x^2 - 4(s+2)(s+1)\}^{1/2}\}/\{(s+2)(s+1)u_s\}, \tag{1}$$

then for $x > 1$, $\{R_s\}$ is monotonic increasing, has the limit $R(x)$ and

$$L_{s+1} < R_{s-1} < R(x). \tag{2}$$

For example, $R(x) > (-x + (9x^2 - 8)^{1/2})/(x^2 - 1)$.

Similarly, if $R(t)$ is the Laplace C.F. for Mills's ratio for the normal integral, and

$$R(t) = \frac{1}{t+}\,\frac{1}{t+}\,\frac{2}{t+}\,\frac{3}{t+}\cdots \qquad, t > 0, \tag{3}$$

with convergents χ_s/ω_s, then [2] with u_s, v_s, w_s defined in terms of the convergents as before, if

$$R_{2s} = (v_{2s} + (2s)!\,(t^2 + 8s + 4)^{1/2})/(2u_{2s}), \tag{4a}$$

and

$$R_{2s+1} = 2w_{2s+1}/\{v_{2s+1} + (2s+1)!\,(t^2 + 8s + 8)^{1/2}\}, \tag{4b}$$

we have convergent monotonic sequences

$$R_0 < R_2 < R_4 \cdots < R < \cdots R_5 < R_3 < R_1,\ t \geqq 0. \tag{5}$$

II. **Irrational Approximants to the Confluent Hypergeometric Function.** With the usual notation, the Gauss C.F. [5] is

$$F(a, 1; c; t) = \frac{1}{1-}\,\frac{b_1 t}{1-}\,\frac{b_2 t}{1-}\cdots,$$