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DIOPHANTINE APPROXIMATION IN A VECTOR SPACE

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1. Throughout this paper, we suppose that S is a real inner product space of dimension at least two and that e is a point of S with unit norm. We denote by S' that subspace of S which has the property that if z belongs to it, then $((z, e)) = 0$, and let u denote a point of S' which has unit norm. For each point z of S, we denote the point $2((z, u))u - z$ by $\bar{z}$ and the point $\bar{z}/\|z\|^2$ by $1/z$. (We assume that there is adjoined to S a "point at infinity" with the usual conventions.) It should be noted that if S is one of E^2, E^3, E^5, and E^9, e is the unit vector with last coordinate 1, and u is the unit vector with first coordinate 1, then $1/z$ restricted to S' reduces to the ordinary reciprocal for real numbers, complex numbers, quaternions, and Cayley numbers, respectively.

Suppose that U is a subset of S' having the following properties:

(i) each element of U is a point of S' with unit norm,

(ii) u belongs to U,

(iii) if x belongs to U, then so do $-x$ and $\bar{x}$,

(iv) if x and y belong to U, then $2((x, y))$ is integral, and

(v) if z is a point of S', there exists a finite sequence $x_1, x_2, \cdots, x_k$, with each term in U, and a finite sequence $n_1, n_2, \cdots, n_k$, with each term an integer, such that $\|z - (n_1x_1 + n_2x_2 + \cdots + n_kx_k)\| < 1$. It is not difficult to see that such a set U exists even when S is infinite dimensional. Notice that when S is one of E^2, E^3, E^5, and E^9 with e and u as above, we may take U to be the set of all units of an appropriate ring of integers.

2. We will now give some definitions which facilitate the statement of the diophantine approximation result below.

A point z of S' is said to be *integral with respect to U* (or *U-integral*) if and only if there exists a finite sequence $x_1, x_2, \cdots, x_k$, with each term in U, and a finite sequence $n_1, n_2, \cdots, n_k$, with each term an integer, such that $z = n_1x_1 + n_2x_2 + \cdots + n_kx_k$. A point z of S' is said to be *rational with respect to U* (or *U-rational*) if and only if there exists a finite sequence $b_0, b_1, b_2, \cdots, b_k$, with each term U-integral, such that z is the value of the continued fraction

$$(2.1) \qquad b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{+ b_k}.$$

A point of S' which is not U-rational is said to be *irrational with respect to U* (or *U-irrational*). If each one of $b_0, b_1, b_2, \cdots, b_k$ is U-integral, we denote by $D(b_0, b_1, b_2, \cdots, b_k)$ the number