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REMARKS ON BASIC CLASSES OF C^∞-FUNCTIONS

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Relationships between analytic functions, basic classes of infinitely differentiable functions, Fourier series, moment problems, and asymptotic series have been investigated for many years (e.g. [1, p. 95]). Recently I established a new relationship [2] which will be described below.

Let $\mathcal{B}_S$ denote the set of Borel subsets of a Borel subset S of the set R of real numbers. Recall that $f: I = [0, 2\pi] \to R$ is a Borel function if $f^{-1}(\mathcal{B}_R) = \{f^{-1}(S) : S \in \mathcal{B}_R\} \subset \mathcal{B}_I$. A Borel function f is bimeasurable if $f(\mathcal{B}_I) \subset \mathcal{B}_R$. Let $\mathcal{P}_S$ denote the set of probability measures μ defined on $\mathcal{B}_S$, let $\mathcal{M}_\mu$ denote the set of μ-measurable subsets of S, and set $\mathcal{U}_S = \bigcap \{\mathcal{M}_\mu : \mu \in \mathcal{P}_S\}$.

The set C^∞ of infinitely differentiable functions defined on I is decomposed into basic classes $C\{M_n\}$ by putting the following growth conditions on successive derivatives: $M_n = (\prod_{j=1}^{n} \mu_j)^{-1}$, where $1 \geqq \mu_1 > \mu_2 > \cdots \to 0$, and $f \in C\{M_n\}$ if there exists $F \in R$ such that $\|f\|_\infty \leqq F$ and $\|f^{(k)}\|_\infty \leqq F^k M_k$, $k = 1, 2, \cdots$. For instance, let C_p denote the class obtained by setting $\mu_n = n^{-p}$, $0 < p < \infty$. Then it is well known that C_1 is the class of analytic functions on I and S. Mandelbrojt showed [4] that $e^{-1/x^2} \in C_2$.

A basic class $C\{M_n\}$ is quasi-analytic if $f \in C\{M_n\}$ implies $f \equiv 0$ when there is a point $x \in I$ such that $0 = f(x) = f^{(k)}(x)$, $k \geqq 1$.

The Denjoy-Carleman theorem asserts that a basic class is quasi-analytic if, and only if, $\sum \mu_k = \infty$; and [2] shows that a basic class is quasi-analytic if, and only if, every function in the class is bimeasurable. Thus, [3] implies (i) if f belongs to a quasi-analytic class, then $f(\mathcal{U}_I) \subset \mathcal{U}_R$ and (ii) assuming the continuum hypothesis $C\{M_n\}$ is quasi-analytic if $f(\mathcal{U}_I) \subset \mathcal{U}_R$ for each $f \in C\{M_n\}$.

For constructing examples it is convenient to have functions f in $C\{M_n\}$ called blips: there exists an interval $[a, b]$ such that $f(x) > 0$ if $x \in (a, b)$ and $f(x) = 0$ otherwise; $[a, b]$ is called the support of f. Clearly no blips with small support exist in a quasi-analytic class. A construction of H. E. Bray [4] can be used to show [2] that every non-quasi-analytic class contains blips with small support; however, the functions so constructed are not analytic blips: they are not analytic on the interiors of their supports. Since it is useful to have analytic blips available, we were led to show that $e^{-1/x^k} \in C_{(1+(1/k))}$, $k = 1, 2, \cdots$.