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SOME RESULTS AND APPLICATIONS ABOUT THE VECTOR ϵ-ALGORITHM

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The ϵ-algorithm is a device found by Wynn [7] to accelerate the convergence of sequences. It is closely related to the Padé table in the following way (Wynn [8]): if we apply the ϵ-algorithm to the partial sums of the power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ then $\epsilon_{2k}^{(n)} = f_{k,n+k}(x)$ where $f_{k,n+k}(x)$ is the Padé approximant to $f(x)$ the denominator of which is of degree k and the numerator of degree $n + k$. Wynn [9] has also proposed a non-scalar ϵ-algorithm working with vectors or matrices or with elements of an associative division algebra over the complex numbers.

There is still no connection between this non-scalar ϵ-algorithm and a Padé table because of non-existence of a theory for the non-scalar Padé table. Yet I think that two recent papers by Wynn [10], [11] are the beginning of such a theory. It is the reason why, in this paper, I should like to speak about convergence theorems for the non-scalar ϵ-algorithm and give an application of the vector ϵ-algorithm to the solution of systems of nonlinear equations.

The non-scalar ϵ-algorithm satisfies the relationship

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + (\epsilon_k^{(n+1)} - \epsilon_k^{(n)})^{-1}$$

with the initial conditions $\epsilon_{-1}^{(n)} = 0$ and $\epsilon_0^{(n)} = S_n$ where the S_n are non-scalar quantities. The inverse of a vector is defined by $z^{-1} = \bar{z}/(z, z)$ where $\bar{z}$ denotes the complex conjugate of z and (z, z) is the scalar product. Let us first give two results concerning the application of the non-scalar ϵ-algorithm to sequences of matrices.

THEOREM 1. *If $S_n = \sum_{k=0}^{n} A^k$ where A is a nonsingular matrix and if $I - A$ is nonsingular, then*

$$\epsilon_2^{(n)} = (I - A)^{-1} \quad \forall n.$$

This theorem is a generalization of a result of Householder [4] for the scalar ϵ-algorithm.

THEOREM 2. *If $\{S_n\}$ is a sequence of square matrices so that*

$$S_{n+1} - S = (A + E_n) \cdot (S_n - S) \text{ or } S_{n+1} - S = (S_n - S) \cdot (A + E_n)$$