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MATRIX INTERPRETATIONS AND APPLICATIONS OF THE CONTINUED FRACTION ALGORITHM

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1. **Introduction.** This paper is concerned with certain aspects of the one to one correspondence between real sequences $\{c_n\}_0^\infty$, formal Laurent series $f(z) = \sum_0^\infty c_n/z^{n+1}$ and infinite Hankel matrices $C = (c_{i+j})_{i,j=0}^\infty$. The finite 'connected' submatrices of C will be denoted by $C_n^{(m)} \equiv (c_{m+i+j})_{i,j=0}^{n-1}$, with $C_n \equiv C_n^{(0)}$, and their determinants by $c_n^{(m)} = \det C_n^{(m)}$.

Also associated with $\{c_n\}$ is the linear functional c^* which acts on the vector space of real polynomials and is determined by $c^*(z^n) = c_n$, $n \geqq 0$. With the ordinary (Cauchy) product of two polynomials $c^*(pq)$ becomes a (Cauchy) bilinear functional on the algebra of real polynomials. If $p(z) = \sum a_i z^i$, $q(z) = \sum b_j z^j$ and $a = (a_0, a_1, a_2, \cdots)^T$, $b = (b_0, b_1, b_2, \cdots)^T$ are the column vectors of coefficients then $c^*(pq) = \sum a_i c_{i+j} b_j = a^T C b$.

The functional $c^*(pq)$ is an *inner product* if and only if $\{c_n\}$, f and C are *positive definite*, that is $c^*(p^2) > 0$ if $p \neq 0$, or equivalently $c_n^{(0)} > 0$ for $n \geqq 0$. An alternative characterization is that $p \neq 0$ and $p(x) \geqq 0$ for $-\infty < x < +\infty$ imply $c^*(p) > 0$. This involves the (unique) decomposition of such a (positive) polynomial p as the sum of two squares of real polynomials whose zeros interlace (strictly) and gives rise to the geometric theory of moment spaces [**20**, **18**, **19**]. If the coefficients $c_n = \int_{-\infty}^{+\infty} t^n d\mu(t)$ are moments of a bounded nondecreasing function μ with infinitely many points of increase then all $c_n^{(2m)} > 0$, since $c^*(t^{2m}p^2) = \int_{-\infty}^{+\infty} t^{2m} \, [p(t)]^2 d\mu(t)$. Conversely if all $c_n^{(0)} > 0$ then the existence of such a μ follows by compactness arguments from the algebraic results to be given below [**12**, **34**, **1**].

2. **Lanczos polynomials.** The algebraic aspects of the theory of orthogonal polynomials carry over to the case in which all $c_n^{(0)} \neq 0$. *Hence this will be assumed.* The material of this section is readily adapted from [**29**, **32**, **11**], for example.

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