ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 2, Spring 1974

BOUNDS FOR MATRIX MOMENTS

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1. **Introduction.** Let A be a real symmetric positive definite $n \times n$ matrix with

$$Au_i = \lambda_i u_i, \quad (i = 1, 2, \cdots, n)$$

$u_i^T u_j = \delta_{ij}$, and $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$. Let r_0 be an arbitrary vector and consider the *Krylov sequence*

$$r_{i+1} = Ar_i, \text{ for } i = 0, 1, \cdots, k-1,$$

so that

$$r_i = A^i r_0 \quad (i = 0, 1, \cdots, k).$$

Let

$$\begin{aligned} \mu_{p,q} = r_p{}^T r_q &= (A^p r_0)^T A^q r_0 \\ &= r_0{}^T A^{p+q} r_0 \\ &\equiv \mu_{p+q}. \end{aligned}$$

Thus if $r_0 = \sum_{i=1}^n \alpha_i u_i$,

$$\mu_m = \sum_{i=1}^m \alpha_i^2 \lambda_i^m \equiv \int \lambda^m \, d\alpha(\lambda) \quad (m = 0, 1, \cdots, 2k)$$

where $\alpha(\lambda) = 0$ for $\lambda \leqq \lambda_1$,

$$\begin{aligned} &= \alpha_1^2 + \cdots + \alpha_t^2 \quad \lambda_t < \lambda \leqq \lambda_{t+1}, \\ &= \alpha_1^2 + \cdots + \alpha_n^2 \quad \lambda_n < \lambda. \end{aligned}$$

Thus, $\{\mu_m\}_{m=1}^{2k}$ are a set of moments associated with the distribution function $\alpha(\lambda)$.

In certain applications (cf. [1]) we are interested in determining bounds for μ_s where s is a positive integer greater than $2k$ or a negative integer. We shall construct algorithms for computing bounds on μ_s where we have an upper bound on the largest eigenvalue and a positive lower bound on the smallest eigenvalue, e.g.,

Received by the editors February 8, 1973.
*This research was in part supported by the AEC and the NSF.

Dedicated to the memory of Professor C. P. Welter (1914–1972).