BOUNDS FOR MATRIX MOMENTS
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1. Introduction. Let $A$ be a real symmetric positive definite $n \times n$ matrix with

$$Au_i = \lambda_i u_i, \quad (i = 1, 2, \cdots, n)$$

$$u_i^T u_j = \delta_{ij}, \quad \text{and} \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$ Let $r_0$ be an arbitrary vector and consider the Krylov sequence

$$r_{i+1} = Ar_i, \quad \text{for} \quad i = 0, 1, \cdots, k - 1,$$

so that

$$r_i = A^i r_0 \quad (i = 0, 1, \cdots, k).$$

Let

$$\mu_{p,q} = r_p^T r_q = (A^p r_0)^T A^q r_0 = r_0^T A^{p+q} r_0 = \mu_{p+q}.$$ Thus if $r_0 = \sum_{i=1}^n \alpha_i u_i$,

$$\mu_m = \sum_{i=1}^m \alpha_i^2 \lambda_i^m = \int \lambda^m \alpha(\lambda) \quad (m = 0, 1, \cdots, 2k)$$

where $\alpha(\lambda) = 0$ for $\lambda \leq \lambda_1$,

$$= \alpha_1^2 + \cdots + \alpha_s^2 \quad \lambda_s < \lambda \leq \lambda_{s+1},$$

$$= \alpha_1^2 + \cdots + \alpha_n^2 \quad \lambda < \lambda_n.$$ Thus, $\{\mu_m\}_{m=1}^{2k}$ are a set of moments associated with the distribution function $\alpha(\lambda)$.

In certain applications (cf. [1]) we are interested in determining bounds for $\mu_s$ where $s$ is a positive integer greater than $2k$ or a negative integer. We shall construct algorithms for computing bounds on $\mu_s$ where we have an upper bound on the largest eigenvalue and a positive lower bound on the smallest eigenvalue, e.g.,