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THE INTERPOLATION OF PICK FUNCTIONS

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Before stating our version of the Cauchy Interpolation Problem it is desirable to recall the definition of the degree of a rational function. Let $f(z)$ be a rational function; then, in a known way, $f(z)$ may be regarded as a continuous map of the Riemann sphere into itself. This mapping has a Brouwer degree, d, which we take to be the degree of the rational $f(z)$. Equivalently, if $f(z)$ is presented as the quotient of two relatively prime polynomials $p(z)$ and $q(z)$, where d' is the algebraic degree of p and d'' is the algebraic degree of q then the degree of $f(z)$ is given by $d = \max(d', d'')$. Finally, we should note that for all but finitely many values of λ the function $f(z) - \lambda$ has exactly d distinct and finite zeros, and these are simple. If it is known that the rational $f(z)$ has degree at most N and that it has at least $N + 1$ zeros, multiplicitly included, then $f(z)$ vanishes identically.

Cauchy Interpolation Problem. Let there be given k distinct interpolation points on the real axis $x_1, x_2, x_3, \cdots, x_k$ and equally many non-negative integers $\nu_1, \nu_2, \nu_3, \cdots, \nu_k$ as well as $N = \sum_{i=1}^{k} (\nu_i + 1)$ real numbers f_{ij} where $1 \leqq i \leqq k$ and $0 \leqq j \leqq \nu_i$. It is required to find a rational function $f(z)$ of degree at most $N/2$ satisfying the N conditions $f^{(j)}(x_i) = f_{ij}$. In any case that we study, the problem will in fact be an interpolation problem: there will be a function $F(z)$, usually not rational, so that the data f_{ij} are obtained from $F^{(j)}(x_i)$.

In the special case when $N = k$, where no derivatives were considered in the problem, the Cauchy Interpolation Problem was exhaustively studied by Löwner in a famous paper [2]. The other extreme case, where $k = 1$, corresponds to the determination of certain Padé approximations of a function, these approximations being on the diagonal or adjacent to the diagonal in the Padé table.

It is important to note that if $f(z)$ is a solution to the Cauchy Interpolation Problem for which the degree of $f(z)$ is strictly smaller than $N/2$ then the solution is unique. Were there another solution $g(z)$, the rational function $f(z) - g(z)$ would have degree at most $N - 1$, but would have at least N zeros, since at each interpolation point x_i there would be a zero of degree $\nu_i + 1$. Thus the difference would vanish identically. We emphasize that this will always be the case when N is odd. It is therefore clear that the Interpolation Problem depends significantly on the parity of N.

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