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VARIATIONAL APPROACH TO THE THEORY OF OPERATOR PADÉ APPROXIMANTS

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I. Definition and Properties of the Operator Padé Approximants. In many applications in theoretical physics, solutions of a problem appear as matrices or operators, and these operators are expressed as formal power series in some parameter (the coupling constant). In many situations the resulting power series are thought to be at best only asymptotic and also the coupling constant may be large, so that even if the series should converge, it may be of limited use. In these circumstances, it is natural to attempt to improve the validity or usefulness of the theory through the use of Padé approximants (P.A.s) even though no mathematical justification for such a step may exist.

We will be concerned with a formal power series

$$T(z) = \sum_{n=0}^{\infty} T_n z^n, \tag{1}$$

where the T_n are matrices or operators in some space. At least two different types of P.A. to $T(z)$ can be considered. The matrix elements of $T(z)$ in some basis may be regarded as a family of power series for each of which a P.A. can be constructed, or a P.A. which is a quotient of polynomial operators can be constructed. The latter approach seems more natural, since it is, as will be shown, basis independent, and it is this approach that will be considered here. In many practical examples, the spaces in question are direct products and intermediate P.A.s, scalar in some indices and operator in others, could be considered. It should be remarked that the individual matrix elements of an operator P.A. may be complicated functions of z; for example, the denominator of the $[M, N]$ P.A. to a matrix in n dimensions is a polynomial of degree nN.

We consider two types of P.A. to T, the right and the left, of the forms

$$\begin{aligned} [M, N]_T(z) &= P_M(z)[Q_N(z)]^{-1} \\ &= \left(\sum_{i=0}^{M} p_i z^i\right)\left(\sum_{j=0}^{N} q_j z^j\right)^{-1}, \end{aligned} \tag{2a}$$

$$[\tilde{M}, \tilde{N}]_T(z) = [\tilde{Q}_N(z)]^{-1}\tilde{P}_M(z)$$

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