ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 1, Winter 1974

UPPER AND LOWER BOUNDS FOR THE NUMBER OF SOLUTIONS OF FUNCTIONAL EQUATIONS INVOLVING k-SET CONTRACTIONS

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1. **Introduction.** In [2] J. Scanlon proved theorems yielding upper and lower bounds on the number of solutions to a functional equation where the function was a compact perturbation of the identity operator. In [7] the author generalized some of the results to get a local version for certain noncompact mappings. The purpose of this paper is to show that these results can be generalized much further. We shall see that the results are true for the k-set contractions studied by F. E. Browder and R. D. Nussbaum in [1] and [5]. Since the work on this paper was done, R. D. Nussbaum has published a similar result. See Appl. Anal. 1 (1971).

2. **Preliminaries.** Let X be a real Banach space. For any subset of X, Ω, define the *measure of compactness* of Ω to be $\gamma(\Omega) = \inf\{d \mid \Omega$ can be covered by a finite number of sets of diameter less than or equal to $d\}$ (See [4] p. 413). We shall say that g is a *k-set contraction* if given any bounded set $A \subset X$, $g(A)$ is a bounded subset of X and $\gamma(g(A)) \leqq k\gamma(A)$ (See [3] and [5]). We shall consider only k-set contractions for which $k < 1$. An example of a k-set contraction is the sum of a compact map and a contraction.

Now suppose that G is an open bounded subset of X and that $f : \bar{G} \to X$ is a k-set contraction for which $a \notin (I - f)(\partial G)$. It is then possible to define the topological degree of $I - f$ at a with respect to G, $d(I - f, G, a)$, as is shown by Nussbaum in [5]. This definition of topological degree has all of the usual properties of topological degree and contains the usual definition of degree.

We next state a lemma giving some properties of k-set contractions that we shall need. Proofs of these properties can be found in [5].

LEMMA 1. *Suppose that f is a differentiable k-set contraction. Then*

(1) *$f'(x)$ is also a k-set contraction,*

(2) *the spectrum of $f'(x)$ is finite for $|\lambda| > k$, and*

(3) *$I - f$ is a Fredholm map of index zero.*

*This work was performed while the author was at the University of Arizona on a post doctoral fellowship which was supported by a Science Developmental Grant from the National Science Foundation.

Received by the Editors February 2, 1972.

AMS (1970) *subject classifications*. Primary 47H15; Secondary 47H10.