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FIXED POINT THEOREMS FOR CONTRACTION MAPPINGS WITH APPLICATIONS TO NONEXPANSIVE AND PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. The principal result states that if H is a closed and convex subset of a Banach space X, G a subset of H open relative to H with the origin 0 in G, and if $U : \bar{G} \to H$ is a contraction mapping satisfying (i) $U(x) = \alpha x$ implies $\alpha \leqq 1$ for $x \neq 0$ in the boundary of G relative to the closed subspace $\mathcal{H}$ of X spanned by H, then U has a fixed point in $\bar{G}$. This result is then used to obtain some fixed point theorems for nonexpansive and pseudo-contractive mappings. It may be compared with a recent result of W. V. Petryshyn for "condensing mappings," a class of mappings more general than the contraction mappings. Petryshyn has shown that if G is bounded with 0 in the interior of G and if $U : \bar{G} \to X$ is a condensing mapping satisfying $U(x) = \alpha x$ implies $\alpha \leqq 1$ for x on the boundary of G, then U has a fixed point in $\bar{G}$. This paper shows that for contraction mappings G need not be bounded, and under certain circumstances, 0 may be on the boundary of G.

1. **Introduction.** Although we restrict our attention in this paper to contraction, nonexpansive, and pseudo-contractive mappings, we will compare our principal results with a recent theorem of W. V. Petryshyn for "condensing mappings." We begin with a description of Petryshyn's result.

For a bounded subset A of a real Banach space X, let $\gamma(A)$ denote the measure of noncompactness of A [17], that is,

$$\gamma(A) = \inf\{d : A \text{ can be covered by a finite number of sets each of diameter} \leqq d\}.$$

Following [22] we say that a continuous mapping $T : G \to X$, $G \subset X$, is *condensing* if for every bounded set $A \subset G$ such that $\gamma(A) \neq 0$ it is the case that $\gamma(T(A)) < \gamma(A)$. This class of mappings includes mappings of the form $T = S + C$ where S is a contraction mapping (i.e., there exists $k < 1$ such that $\|S(x) - S(y)\| \leqq k\|x - y\|$ for all x, $y \in G$) and C a compact mapping of G into X.

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