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INDICES OF LINDELÖF FUNCTIONS AND THEIR DERIVATIVES

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1. **Introduction.** A transcendental entire function $f(z)$ is said to be of bounded index if there exists an integer N, independent of z, such that

$$\max_{0 \leqq k \leqq N} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\} \geqq \frac{|f^{(j)}(z)|}{j!} \tag{1.1}$$

holds for all z and j. The least such integer N is called the index of f (cf. [3], [8]). It is known [11] that a function of bounded index is at most exponential type but all functions of exponential type need not be of bounded index (see [11], [13]). Lee and Shah [6], [7] have shown that if $\{a_n\}$ is any sequence of positive numbers such that $a_{n+1}/a_n \geqq \gamma > 1$, and a and b are any complex numbers, then

$$F(z) = e^{az+b} \prod_1^\infty \{1 - z/a_n\}$$

and all successive derivatives $F^{(k)}(z)$ are of bounded index. Further if $\{a_n\}$ is any sequence of complex numbers such that $|a_{n+1}| \geqq 5^n |a_n|$, $|a_1| \geqq 5$, then $\psi(z) = \prod_1^\infty (1 - z/a_n)$ and all derivatives $\psi^{(k)}(z)$ are of bounded index [10]. (The first author has proved this result with "5" replaced by "4" in her doctoral dissertation.)

In this paper we investigate the index of the Lindelöf function, f, [9], [4] defined by

$$f(z) = \prod_{n=1}^\infty (1 - z/n^\alpha), \qquad \alpha > 1. \tag{1.2}$$

Pugh (cf. [10, p. 192]) has shown that if $\alpha \geqq 8$, then f is of bounded index. We prove here

THEOREM 1. *Let* $f(z) = f(z, \alpha) = \prod_1^\infty (1 - z/n^\alpha)$, $\alpha > 1$; *then* $f(z)$ *is of bounded index. It is of index one if* $\alpha \geqq 3$.

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