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PRESERVATION OF COPRODUCTS BY $\mathrm{Hom}_R(M, -)$

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The functor $\mathrm{Hom}_R(M, -)$ from the category of left R-modules into the category of abelian groups always preserves products but preserves coproducts only in special cases. An obvious sufficient condition for the preservation of coproducts is that M be finitely generated. In several significant special cases (for example, when M is projective or R is left Noetherian) finite generation is also necessary. H. Bass has stated [**1**, p. 54] that finite generation is not in general necessary for the preservation of coproducts and he has given a necessary and sufficient condition which we state in slightly altered form: $\mathrm{Hom}_R(M, -)$ *preserves coproducts if and only if M is not the union of any nest of proper submodules of the form* $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ *(i a positive integer).* In this note we present a simple example of a non-finitely generated module M for which $\mathrm{Hom}_R(M, -)$ preserves coproducts and we discuss the effect of some additional hypotheses on coproduct preservation.

We make four assumptions that hold throughout this note: R is a ring with identity. All modules are unitary left R-modules. A map is an R-homomorphism. N is the set of positive integers.

THEOREM. *There exists a Boolean ring $\boldsymbol{R}$ which has cardinal $\aleph_1$ and contains a maximal ideal $\boldsymbol{M}$ which is neither finitely nor countably generated but for which* $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{M}, -)$ *preserves coproducts.*

PROOF. For each ordinal number β let S_β be the set of all ordinals α such that $\alpha < \beta$. Let Ω be the least ordinal of uncountable cardinal. The validity of our example will be seen to stem from the following fact: A subset X of S_Ω is cofinal (i.e., for every $\alpha \in S_\Omega$ there is a $\beta \in X$ such that $\alpha < \beta$) if and only if it is uncountable.

Let $\boldsymbol{R}$ be the subring of the ring of all subsets of S_Ω that is generated by the set of all 'segments' $\{S_\alpha \mid \alpha \leqq \Omega\}$. Then $\boldsymbol{R}$ is a Boolean ring with identity S_Ω and has cardinal $\aleph_1$. Let $\boldsymbol{M}$ be the ideal of $\boldsymbol{R}$ generated by the set of all 'short' segments $\{S_\alpha \mid \alpha < \Omega\}$. Then $\boldsymbol{M}$ is proper and maximal. Let $A_i \in \boldsymbol{M}$ ($i \in N$). For each i in N we have an $\alpha(i) < \Omega$ such that $A_i \subseteq S_{\alpha(i)}$. Since $\{\alpha(i) \mid i \in N\}$ is countable (= not cofinal),

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