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LAMBERT SERIES, FALSE THETA FUNCTIONS, AND PARTITIONS

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1. **Introduction.** One of the recent important results in the theory of partitions is the following theorem due to B. Gordon [5].

THEOREM. Let $A_{k,a}(N)$ *denote the number of partitions of* N *into parts* $\not\equiv 0, \pm a \pmod{2k+1}$. *Let* $B_{k,a}(N)$ *denote the number of partitions of* N *of the form* $N = \sum_{i=1}^{\infty} f_i i$ (f_i *denotes the number of times the summand* i *appears in the partition*) *where* $f_1 \leqq a - 1$ *and* $f_i + f_{i+1} \leqq k - 1$. *Then* $A_{k,a}(N) = B_{k,a}(N)$.

This theorem reduces to the Rogers-Ramanujan identities when $k = 2$.

In this paper we shall study a partition function $W_{k,i}(n; N)$ which is somewhat similar to $B_{k,i}(N)$. $W_{k,i}(n; N)$ denotes the number of partitions of N of the form $N = \sum_{i=1}^{n} f_i i$, with $f_1 = i$, $f_j \leqq k - 1$, and $f_j + f_{j+1} = k$ or $k + 1$ for $1 \leqq j \leqq n - 1$. We let $w_{k,i}(n; q) = 1 + \sum_{N=1}^{\infty} W_{k,i}(n; N) q^N$. Our first result relates $w_{k,i}(n; q)$ to certain Lambert series.

THEOREM 1. *For* $|q| < 1$,

$$1 - \sum_{n=1}^{\infty} q^{(2k-1)n^2/2 + n/2 - (k-i)n} \frac{(1 - q^{2n(k-i)})}{1 + q^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n w_{k,i}(n; q)}{(1+q)(1+q^2) \cdots (1+q^n)}.$$

When $i = k - 1$, we see that the left-hand series in Theorem 1 reduces to a false theta series. From Theorem 1 it is possible to prove results on partitions which we shall examine in §3.

2. **Proof of Theorem 1.** We define the function $f_{k,i}(x)$ as follows:

$$(2.1) \qquad f_{k,i}(x) = \sum_{n=0}^{\infty} x^{kn} q^{(2k-1)n^2/2 + n/2 - in} (1 - x^i q^{2ni}) \frac{(-1)_n}{(-xq)_n},$$

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