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ON THE HURWITZ ZETA-FUNCTION

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1. **Introduction.** Briggs and Chowla [2] and Hardy [3] calculated the coefficients of the Laurent expansion of the Riemann zeta-function $\zeta(s)$ about $s = 1$. Kluyver [4] found a certain infinite series representation for these coefficients. In another paper [1] Briggs found estimates for the coefficients. These estimates were improved by Lammel [6]. Using these estimates, Lammel also gave a simple proof of the fact that $\zeta(s)$ has no zeros on $|s - 1| \leqq 1$.

Using the same technique as in [2] and [6], we derive expressions for the coefficients of the Laurent expansion of the generalized or Hurwitz zeta-function $\zeta(s, a)$, $0 < a \leqq 1$, about $s = 1$. A similar formula for these coefficients has been given by Wilton [11]. We then obtain estimates for these coefficients. Our technique here is somewhat simpler than in [6], and as a special case we obtain improved estimates for the Laurent coefficients of $\zeta(s)$. Next, we use our estimates to show that $\zeta(s, a) - a^{-s}$ has no zeros on $|s - 1| \leqq 1$. We conclude by indicating a new, simple proof of a representation formula for $\zeta(s, a)$ that was first discovered by Hurwitz.

2. **Calculation of the Laurent coefficients.** In the sequel we shall need a slightly different version of the Euler-Maclaurin summation formula from what is usually given. Let $f \in C^n$ on $[\alpha, m]$, where m is an integer. Then,

$$\sum_{\alpha < k \leqq m} f(k) = \int_\alpha^m f(x)dx + \sum_{k=1}^{n} (-1)^k \frac{B_k}{k!} f^{(k-1)}(m) + \sum_{k=1}^{n} (-1)^{k+1} P_k(\alpha) f^{(k-1)}(\alpha) + R_n, \tag{2.1}$$

where

$$R_n = (-1)^{n+1} \int_\alpha^m P_n(x) f^{(n)}(x)dx.$$

Here, B_k, $1 \leqq k \leqq n$, denotes the kth Bernoulli number, and $P_k(x)$,

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