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SOME COUNTEREXAMPLES INVOLVING SELFADJOINT OPERATORS

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1. **Introduction.** We present several counterexamples related to the convergence, (generalized) addition, and (generalized) commutation of (unbounded) skew-adjoint operators.

2. **Convergence of skew-adjoint operators.** Let A_n $(n = 0, 1, 2, \cdots)$ be a skew-adjoint operator on a Hilbert space $\mathcal{H}$. We say that A_n *converges to* A_0 and we write $\lim_{n\to\infty} A_n = A_0$ iff

$$\lim_{n\to\infty} (\lambda I - A_n)^{-1}f = (\lambda I - A_0)^{-1}f \tag{1}$$

for all $f \in \mathcal{H}$ and all $\lambda \in \mathbf{R}\setminus\{0\}$ ($\mathbf{R}$ is the real line and I is the identity on $\mathcal{H}$). This is equivalent to

$$\lim_{n\to\infty} U_n(t)f = U_0(t)f \tag{2}$$

for all $t \in \mathbf{R}$ and all $f \in \mathcal{H}$ where $U_n = \{U_n(t); t \in \mathbf{R}\}$ is the (C_0) unitary group generated by A_n, $n = 0, 1, 2, \cdots$. The above result is an immediate consequence of Stone's theorem and the Trotter-Neveu-Kato approximation theorem for (C_0) semigroups of operators (cf. for instance Goldstein [**5**], Kato [**6**], Yosida [**9**]).

A useful sufficient condition for (1) to hold is given by the following well-known simple result.

LEMMA 1. *Let A_n be skew-adjoint operators on $\mathcal{H}$, $n = 0, 1, 2, \cdots$. Then* (1) *holds for all $f \in \mathcal{H}$ and all $\lambda \in \mathbf{R}\setminus\{0\}$ if there is a subspace $\mathfrak{D} \subset \mathfrak{D}(A_0)$ (= the domain of A_0) such that*

(i) *A_0 is the closure of $A_0 \mid \mathfrak{D}$,*

(ii) *for all $f \in \mathfrak{D}$, $f \in \mathfrak{D}(A_n)$ for n sufficiently large and $\lim_{n\to\infty} A_n f = A_0 f$.*

Our first example shows that the sufficient condition given in Lemma 1 is far from being necessary.

EXAMPLE 1. *There is a sequence U_n $(n = 0, 1, 2, \cdots)$ of (C_0)*

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