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CALCULUS OF VARIATIONS IN COMPLEX VECTOR BUNDLES

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1. **Introduction.** The theory of the calculus of variations, and in particular Jacobi vector fields, is a very important tool in the study of the topology of Riemannian manifolds. Recently Goldberg and Kobayashi [1] have introduced the notion of holomorphic bisectional curvature of a Kähler manifold. Actually this notion makes sense in an arbitrary almost Hermitian vector bundle (see §2). In fact in a slightly different form the notion of holomorphic bisectional curvature has been used by several authors [3], [5], [6] to define notions of positivity of holomorphic vector bundles which generalize that of holomorphic line bundles.

In this paper we define and study a calculus of variations in almost Hermitian vector bundles via holomorphic bisectional curvature. The basis of our discussion is the following observation. Let M be a Kähler manifold with almost complex structure J. If σ is a geodesic in M and Y is a Jacobi vector field along σ, then JY is not a Jacobi vector field. Moreover if $X = Y + JY$ then X satisfies the differential equation

$$2X'' = JR_{\sigma' J\sigma'}X, \tag{1.1}$$

where R denotes the curvature operator of M. Now (1.1) makes sense in an arbitrary almost Hermitian vector bundle with a given covariant derivative. This is the starting point for our theory of the calculus of variations. We obtain new versions of several well-known theorems in the calculus of variations of a Riemannian manifold, including Myer's theorem and the Rauch comparison theorem. (We do not discuss the Morse index theorem, because versions of it already are known in the context of vector bundles, e.g., [7].) A defect in our theory is that there seems to be no good substitute for the exponential map of Riemannian geometry, although we define the notion of conjugate point.

In §2 we describe analogues for an almost Hermitian bundle E of Jacobi vector fields and the index form. We develop some properties of the index form in §3 and prove a version of Myer's theorem for

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