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SOME PROBABILISTIC REMARKS ON FERMAT'S LAST THEOREM

P. ERDÖS AND S. ULAM

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers satisfying $a_n = (c + o(1))n^\alpha$ for some $\alpha > 1$. One can ask: Is it likely that $a_i + a_j = a_r$ or, more generally, $a_{i_1} + \cdots + a_{i_n} = a_{i_r}$ has infinitely many solutions. We will formulate this problem precisely and show that if $\alpha > 3$ then with probability 1, $a_i + a_j = a_r$ has only finitely many solutions, but for $\alpha \leqq 3$, $a_i + a_j = a_r$ has with probability 1 infinitely many solutions. Several related questions will also be discussed.

Following [1] we define a measure in the space of sequences of integers. Let $\alpha > 1$ be any real number. The measure of the set of sequences containing n has measure $c_1 n^{1/\alpha - 1}$ and the measure of the set of sequences not containing n has measure $1 - c_1 n^{1/\alpha - 1}$. It easily follows from the law of large numbers (see [1]) that for almost all sequences $A = \{a_1 < a_2 < \cdots\}$ ("almost all" of course, means that we neglect a set of sequences which has measure 0 in our measure) we have

$$A(x) = (1 + o(1))c_1 \sum_{n=1}^{x} \frac{1}{n^{1/\alpha - 1}} = (1 + o(1))c_1 \alpha x^{1/\alpha} \tag{1}$$

where $A(x) = \sum_{a_i < x} 1$. (1) implies that for almost all sequences A

$$a_n = (1 + o(1))(n/c_1\alpha)^\alpha. \tag{2}$$

Now we prove the following

THEOREM. *Let $\alpha > 3$. Then for almost all A*

$$a_i + a_j = a_r \tag{3}$$

has only a finite number of solutions. If $\alpha \leqq 3$, then for almost all A, (3) *has infinitely many solutions.*

It is well known that $x^3 + y^3 = z^3$ has no solutions, thus the sequence $\{n^3\}$ belongs to the exceptional set of measure 0.

Assume $\alpha > 3$. Denote by E_α the expected number of solutions of $a_i + a_j = a_r$. We show that E_α is finite and this will immediately

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