ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 4, Fall 1971

GENERALIZATIONS OF MIDPOINT RULES

ROBERT E. BARNHILL

ABSTRACT. A midpoint rule proposed by Jagermann and improved upon by Stetter is generalized to Hermite-type quadrature rules and to first degree cubature rules. Remainder terms are included in both cases.

1. **Introduction.** This note contains two types of generalizations of a midpoint rule proposed by Jagermann [1] and improved upon by Stetter [3]. The first generalization involves a Hermite type of midpoint rule and is discussed in §2. The second generalization concerns cubature rules for a function of two variables and is in §3. In both cases, error terms are included, from which asymptotic estimates can be derived.

2. **Hermite-type midpoint rules.** The integral to be approximated is $\int_a^b p(x)f(x)dx$, where $p(x) \geqq 0$, $p(x)$ does not vanish identically on any subinterval of $[a, b]$, and $\int_a^b p(x)dx = 1$, Stetter [3] has proved the following:

Let $N \geqq 1$ and

$$S_N(f) \equiv \int_a^b p(x)f(x)dx - \frac{1}{N}\sum_{i=0}^{N-1} f(a_i),$$

where $a_i = N\int_{x_i}^{x_{i+1}} tp(t)dt$, $i = 0, 1, \cdots, N-1$, and the x_i, $a = x_0 < x_1 < \cdots < x_N = b$, are chosen so that $1/N = \int_{x_i}^{x_{i+1}} p(x)dx$. Then $S_N(f) = \frac{1}{2} S_N(x^2)f''(\epsilon)$, $a < \epsilon < b$.

We generalize this theorem as follows:

THEOREM 1. *Let*

$$R_N{}^{(1)}(f) \equiv \int_a^b p(x)f(x)dx - \left[\frac{1}{N}\sum_{i=0}^{N-1} f(a_i) + \sum_{i=0}^{N-1} E_i(x)f'(a_i)\right],$$

where $p(x)$ is as above,

$$E_i(x) = \int_{x_i}^{x_{i+1}} xp(x)dx - a_i/N$$

and the a_i are chosen so that

$$\int_{x_i}^{x_{i+1}} p(x)(x - a_i)^2 dx = 0.$$

Received by the editors April 8, 1970.
AMS 1970 *subject classifications.* Primary 41A55, 65D30.