ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 3, Summer 1971

REGULARITY THEOREMS AND GERŠGORIN THEOREMS FOR MATRICES OVER RINGS WITH VALUATION

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ABSTRACT. The collection of root-location theorems for matrices of complex numbers is now quite extensive. Since their proofs involve chiefly manipulation of absolute value inequalities, many of these theorems can be extended to noncommutative domains, in particular to quaternion matrices. Secondly, the ring of polynomials has a valuation with properties that differ slightly from those of the ordinary absolute value function. Using this valuation, a different type of regularity theorem is obtainable. With a suitable definition of proper value of a matrix of polynomials, these regularity theorems also lead to root-location theorems. Finally, bounds for determinants can be obtained. These bounds are given in terms of the valuation: for polynomials, they are bounds on the degree.

1. **Introduction.** Let $A = [a_{ij}]_1^n$ be a matrix of complex numbers. The curtate row and column-sums $R_{i,p}$, $C_{i,p}$ are defined by

$$R_{i,p}^p = \sum_{j \neq i} |a_{ij}|^p, \qquad C_{i,p}^p = \sum_{j \neq i} |a_{ji}|^p.$$

Thus $R_i \equiv R_{i,1}$ is the sum of the absolute values of the nondiagonal elements of the ith row; $C_i \equiv C_{i,1}$ is the sum of the absolute values of the nondiagonal elements of the ith column. A is called regular if no nonzero vector x exists such that $Ax = 0$. If A is regular, so is A^T, its transpose; and A is invertible. The constants used below are real and subject to $0 \leqq \alpha, \beta, \gamma \leqq 1$; $1 \leqq p, p', p''$; $\alpha + \beta + \gamma = 1$;

$$p^{-1} + q^{-1} = p'^{-1} + q'^{-1} = p''^{-1} + q''^{-1} = 1;$$
$$0 \leqq k_i, \quad \sum (k_i + 1)^{-1} \leqq 1.$$

$R_{i,\infty} = m_i$ is the maximum of the absolute values of the nondiagonal elements of the ith row of A; $C_{i,\infty} = c_i$.

The following hypotheses are known to guarantee the nonsingularity of A.

(1.1) $$\forall_i \{|a_{ii}| > R_i\} \qquad [8].$$

(1.2) $$\forall_i \{|a_{ii}| > C_i\}.$$

(1.3) $$\forall_i \{|a_{ii}| > R_i^\alpha C_i^{1-\alpha}\} \qquad [15].$$

Received by the editors September 5, 1969 and, in revised form, March 4, 1970.

AMS 1970 subject classifications. Primary 15A18, 15A33; Secondary 15A09, 15A15.