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ON LIAPUNOV'S DIRECT METHOD

JAMES S. MULDOWNEY

We shall consider the system of ordinary differential equations

$$x' = f(t, x), \qquad t \in [0, \infty), \quad x \in D, \tag{1}$$

where D is an open connected subset of R^n containing the zero vector and f is a function from $[0, \infty) \times D$ to R^n such that solutions to (1) exist locally in the Carathéodory sense (cf. [4, p. 42]). *We denote by* $\mathcal{L}(1)$ *the class of real-valued functions* $V(t, x)$ *on* $[0, \infty) \times D$ *such that* $V(t, x(t))$ *is nonincreasing whenever* $x(t)$ *is a solution of* (1). A sufficient condition for $V \in \mathcal{L}(1)$ is that V be continuous in (t, x), locally Lipschitzian in x and satisfy

$$\limsup_{h \to 0+} [V(t + h, x + hf(t, x)) - V(t, x)]/h \leqq 0$$

for all (t, x), when f is continuous (cf. [**14**, p. 4]).

All of the applications of Liapunov's direct method with which we are here concerned are based on the observation that if $V \in \mathcal{L}(1)$ and (t_0, x_0), (t_1, x_1) are such that $t_0 < t_1$ and $V(t_0, x_0) < V(t_1, x_1)$ then there is no solution $x(t)$ of (1) such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

Notation. (i) A solution $x(t)$ such that $x(t_0) = x_0$ will often be denoted $x(t; t_0, x_0)$.

(ii) If $x_0 \in R^n$, $r \in (0, \infty)$, then $B(x_0, r) = \{x : |x - x_0| < r\}$, where $|\ |$ denotes any norm.

(iii) $2^D = \{X : X \subset D\}$.

(iv) Let $x, y \in R^n$:

$$\rho(x, y) = |x - y|, \quad \text{if } x \neq \infty,\ y \neq \infty;$$

$$\rho(x, y) = \frac{1}{|x|}, \quad \text{if } y = \infty;$$

$$\rho(x, X) = \inf \{\rho(x, y) : y \in X\}, \quad \text{if } X \subset R^n$$

and $x \to X$ means $\rho(x, X) \to 0$.

(v) If $X \subset R^n$ then $\bar{X}$ and ∂X denote the closure and boundary of X respectively.

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