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A METRIC FOR WEAK CONVERGENCE OF DISTRIBUTION FUNCTIONS[1]

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Let Δ denote the set of distribution functions, that is, left-continuous nondecreasing functions from the real line into $[0, 1]$. The set of distribution functions of random variables (functions in Δ with sup 1 and inf 0) will be denoted by Δ_{rv}. The following facts are well known (see, e.g., [2]).

1. The space Δ is sequentially compact with respect to weak convergence. That is, any sequence of functions in Δ has a weakly convergent subsequence (Helly's First Theorem). However, Δ_{rv} does not have this property.

2. The set Δ is a metric space under the Lévy metric L defined for any F, G in Δ by

$$L(F, G) = \inf\{h;\ F(x - h) - h \leqq G(x) \leqq F(x + h) + h \text{ for all } x\}.$$

3. If (F_n) is a sequence in Δ_{rv} and F is also in Δ_{rv} then F_n converges weakly to F iff $L(F_n, F) \to 0$. Thus for sequences in Δ_{rv} whose limit is also in Δ_{rv} weak convergence and convergence in the L-metric are equivalent. The hypothesis that the limit belong to Δ_{rv} is necessary, for there are sequences in Δ_{rv} which converge weakly to a limit in Δ but do not converge in the Lévy metric. (One such sequence is discussed in this paper.)

Statement 3 shows that the relationship between weak convergence (in the sense of Helly's First Theorem) and convergence in the metric L is unsatisfactory. This state of affairs is due to the fact that the Lévy metric is sensitive to what happens at $+\infty$ and $-\infty$, while weak convergence is not. The purpose of this paper is to show that a modification of the Lévy metric yields a metric for Δ for which convergence corresponds precisely to weak convergence.

For any F, G in Δ and $h > 0$, define the properties

(1) $A(F, G; h)$ iff $F(x - h) - h \leqq G(x)$ for $-1/h < x < 1/h + h$,

(2) $B(F, G; h)$ iff $F(x + h) + h \geqq G(x)$ for $-h - 1/h < x < 1/h$,

and let

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